CHAPTER

10

Return and Risk

The Capital Asset Pricing Model (CAPM)

Expected returns on common stocks can vary quite a bit. One important determinant is the industry in which a company operates. For example, according to recent estimates from Ibbotson Associates, the median expected return for department stores, which includes companies such as Sears and Kohls, is 11.63 percent, whereas computer service companies such as Microsoft and Oracle have a median expected return of 15.46 percent. Air transportation companies such as Delta and Southwest have a median expected return that is even higher: 17.93 percent.

These estimates raise some obvious questions. First, why do these industry expected returns differ so much, and how are these specific numbers calculated? Also, does the higher return offered by airline stocks mean that investors should prefer these to, say, department store stocks? As we will see in this chapter, the Nobel Prize–winning answers to these questions form the basis of our modern understanding of risk and return.

10.1 Individual Securities

In the first part of Chapter 10, we will examine the characteristics of individual securities. In particular, we will discuss:

1. *Expected return*: This is the return that an individual expects a stock to earn over the next period. Of course, because this is only an expectation, the actual return may be either higher or lower. An individual's expectation may simply be the average return per period a security has earned in the past. Alternatively, it may be based on a detailed analysis of a firm's prospects, on some computer-based model, or on special (or inside) information.
2. *Variance and standard deviation*: There are many ways to assess the volatility of a security's return. One of the most common is variance, which is a measure of the squared deviations of a security's return from its expected return. Standard deviation is the square root of the variance.
3. *Covariance and correlation*: Returns on individual securities are related to one another. Covariance is a statistic measuring the interrelationship between two securities. Alternatively, this relationship can be restated in terms of the correlation between the two securities. Covariance and correlation are building blocks to an understanding of the beta coefficient.

10.2 Expected Return, Variance, and Covariance

Expected Return and Variance

Suppose financial analysts believe that there are four equally likely states of the economy: depression, recession, normal, and boom. The returns on the Supertech Company are expected to follow the economy closely, while the returns on the Slowpoke Company are not. The return predictions are as follows:

	Supertech Returns R_{At}	Slowpoke Returns R_{Bt}
Depression	−20%	5%
Recession	10	20
Normal	30	−12
Boom	50	9

Variance can be calculated in four steps. An additional step is needed to calculate standard deviation. (The calculations are presented in Table 10.1.) The steps are these:

1. Calculate the expected return:

   Supertech

   $$\frac{-0.20 + 0.10 + 0.30 + 0.50}{4} = 0.175 = 17.5\% = \bar{R}_A$$

   Slowpoke

   $$\frac{0.05 + 0.20 - 0.12 + 0.09}{4} = 0.055 = 5.5\% = \bar{R}_B$$

2. For each company, calculate the deviation of each possible return from the company's expected return given previously. This is presented in the third column of Table 10.1.
3. The deviations we have calculated are indications of the dispersion of returns. However, because some are positive and some are negative, it is difficult to work with them in this form. For example, if we were to simply add up all the deviations for a single company, we would get zero as the sum.

   To make the deviations more meaningful, we multiply each one by itself. Now all the numbers are positive, implying that their sum must be positive as well. The squared deviations are presented in the last column of Table 10.1.
4. For each company, calculate the average squared deviation, which is the variance:[1]

   Supertech

   $$\frac{0.140625 + 0.005625 + 0.015625 + 0.105625}{4} = 0.066875$$

[1] In this example, the four states give rise to four *possible* outcomes for each stock. Had we used past data, the outcomes would have actually occurred. In that case, statisticians argue that the correct divisor is $N - 1$, where N is the number of observations. Thus the denominator would be 3 [= (4 − 1)] in the case of past data, not 4. Note that the example in Section 9.5 involved past data and we used a divisor of $N - 1$. While this difference causes grief to both students and textbook writers, it is a minor point in practice. In the real world, samples are generally so large that using N or $N - 1$ in the denominator has virtually no effect on the calculation of variance.

Table 10.1
Calculating Variance and Standard Deviation

(1) State of Economy	(2) Rate of Return	(3) Deviation from Expected Return	(4) Squared Value of Deviation
	Supertech*	**(Expected return = 0.175)**	
	R_{At}	$(R_{At} - \bar{R}_A)$	$(R_{At} - \bar{R}_A)^2$
Depression	−0.20	−0.375	0.140625
		(= −0.20 − 0.175)	[= (−0.375)^2]
Recession	0.10	−0.075	0.005625
Normal	0.30	0.125	0.015625
Boom	0.50	0.325	0.105625
			0.267500
	Slowpoke†	**(Expected return = 0.055)**	
	R_{Bt}	$(R_{Bt} - \bar{R}_B)$	$(R_{Bt} - \bar{R}_B)^2$
Depression	0.05	−0.005	0.000025
		(= 0.05 − 0.055)	[= (−0.005)^2]
Recession	0.20	0.145	0.021025
Normal	−0.12	−0.175	0.030625
Boom	0.09	0.035	0.001225
			0.052900

$$*\bar{R}_A = \frac{-0.20 + 0.10 + 0.30 + 0.50}{4} = 0.175 = 17.5\%$$

$$\text{Var}(R_A) = \sigma_A^2 = \frac{0.2675}{4} = 0.066875$$

$$\text{SD}(R_A) = \sigma_A = \sqrt{0.066875} = 0.2586 = 25.86\%$$

$$\dagger\bar{R}_B = \frac{0.05 + 0.20 - 0.12 - 0.09}{4} = 0.055 = 5.5\%$$

$$\text{Var}(R_B) = \sigma_B^2 = \frac{0.0529}{4} = 0.013225$$

$$\text{SD}(R_B) = \sigma_B = \sqrt{0.013225} = 0.1150 = 11.50\%$$

Slowpoke

$$\frac{0.000025 + 0.021025 + 0.030625 + 0.001225}{4} = 0.013225$$

Thus, the variance of Supertech is 0.066875, and the variance of Slowpoke is: 0.013225.

5. Calculate standard deviation by taking the square root of the variance:

Supertech

$$\sqrt{0.066875} = 0.2586 = 25.86\%$$

Slowpoke

$$\sqrt{0.013225} = 0.1150 = 11.50\%$$

Algebraically, the formula for variance can be expressed as:

$$\text{Var}(R) = \text{Expected value of } (R - \bar{R})^2$$

where $\bar{R}$ is the security's expected return and R is the actual return.

A look at the four-step calculation for variance makes it clear why it is a measure of the spread of the sample of returns. For each observation we square the difference between the actual return and the expected return. We then take an average of these squared differences. Squaring the differences makes them all positive. If we used the differences between each return and the expected return and then averaged these differences, we would get zero because the returns that were above the mean would cancel the ones below.

However, because the variance is still expressed in squared terms, it is difficult to interpret. Standard deviation has a much simpler interpretation, which was provided in Section 9.5. Standard deviation is simply the square root of the variance. The general formula for the standard deviation is:

$$\text{SD}(R) = \sqrt{\text{Var}(R)}$$

Covariance and Correlation

Variance and standard deviation measure the variability of individual stocks. We now wish to measure the relationship between the return on one stock and the return on another. Enter **covariance** and **correlation**.

Covariance and correlation measure how two random variables are related. We explain these terms by extending the Supertech and Slowpoke example.

EXAMPLE 10.1

Calculating Covariance and Correlation We have already determined the expected returns and standard deviations for both Supertech and Slowpoke. (The expected returns are 0.175 and 0.055 for Supertech and Slowpoke, respectively. The standard deviations are 0.2586 and 0.1150, respectively.) In addition, we calculated the deviation of each possible return from the expected return for each firm. Using these data, we can calculate covariance in two steps. An extra step is needed to calculate correlation.

1. For each state of the economy, multiply Supertech's deviation from its expected return and Slowpoke's deviation from its expected return together. For example, Supertech's rate of return in a depression is −0.20, which is −0.375 (=−0.20 − 0.175) from its expected return. Slowpoke's rate of return in a depression is 0.05, which is −0.005 (=0.05 − 0.055) from its expected return. Multiplying the two deviations together yields 0.001875 [=(−0.375) × (−0.005)]. The actual calculations are given in the last column of Table 10.2. This procedure can be written algebraically as:

$$(R_{At} - \overline{R}_A) \times (R_{Bt} - \overline{R}_B) \tag{10.1}$$

   where R_{At} and R_{Bt} are the returns on Supertech and Slowpoke in state t. $\overline{R}_A$ and $\overline{R}_B$ are the expected returns on the two securities.

2. Calculate the average value of the four states in the last column. This average is the covariance. That is:[2]

$$\sigma_{AB} = \text{Cov}(R_A, R_B) = \frac{-0.0195}{4} = -0.004875$$

Note that we represent the covariance between Supertech and Slowpoke as either $\text{Cov}(R_A, R_B)$ or σ_{AB}. Equation 10.1 illustrates the intuition of covariance. Suppose Supertech's return is generally above its average when Slowpoke's return is above its average, and Supertech's return is generally

(continued)

[2]As with variance, we divided by N (4 in this example) because the four states give rise to four *possible* outcomes. However, had we used past data, the correct divisor would be $N - 1$ (3 in this example).

Table 10.2 Calculating Covariance and Correlation

State of Economy	Rate of Return of Supertech R_{At}	Deviation from Expected Return $(R_{At} - \bar{R}_A)$	Rate of Return of Slowpoke R_{Bt}	Deviation from Expected Return $(R_{Bt} - \bar{R}_B)$	Product of Deviations $(R_{At} - \bar{R}_A) \times (R_{Bt} - \bar{R}_B)$
		(Expected return = 0.175)		(Expected return = 0.055)	
Depression	−0.20	−0.375 (= −0.20 − 0.175)	0.05	−0.005 (= 0.05 − 0.055)	0.001875 (= −0.375 × −0.005)
Recession	0.10	−0.075	0.20	0.145	−0.010875 (= −0.075 × 0.145)
Normal	0.30	0.125	−0.12	−0.175	−0.021875 (= 0.125 × −0.175)
Boom	0.50	0.325	0.09	0.035	0.011375 (= 0.325 × 0.035)
	0.70		0.22		−0.0195

$$\sigma_{AB} = \text{Cov}(R_A, R_B) = \frac{-0.0195}{4} = -0.004875$$

$$\rho_{AB} = \text{Corr}(R_A, R_B) = \frac{\text{Cov}(R_A, R_B)}{\text{SD}(R_A) \times \text{SD}(R_B)} = \frac{-0.004875}{0.2586 \times 0.1150} = -0.1639$$

below its average when Slowpoke's return is below its average. This shows a positive dependency or a positive relationship between the two returns. Note that the term in Equation 10.1 will be *positive* in any state where both returns are *above* their averages. In addition, 10.1 will still be *positive* in any state where both terms are *below* their averages. Thus a positive relationship between the two returns will give rise to a positive value for covariance.

Conversely, suppose Supertech's return is generally above its average when Slowpoke's return is below its average, and Supertech's return is generally below its average when Slowpoke's return is above its average. This demonstrates a negative dependency or a negative relationship between the two returns. Note that the term in Equation 10.1 will be *negative* in any state where one return is above its average and the other return is below its average. Thus a negative relationship between the two returns will give rise to a negative value for covariance.

Finally, suppose there is no relationship between the two returns. In this case, knowing whether the return on Supertech is above or below its expected return tells us nothing about the return on Slowpoke. In the covariance formula, then, there will be no tendency for the deviations to be positive or negative together. On average, they will tend to offset each other and cancel out, making the covariance zero.

Of course, even if the two returns are unrelated to each other, the covariance formula will not equal zero exactly in any actual history. This is due to sampling error; randomness alone will make the calculation positive or negative. But for a historical sample that is long enough, if the two returns are not related to each other, we should expect the covariance to come close to zero.

The covariance formula seems to capture what we are looking for. If the two returns are positively related to each other, they will have a positive covariance, and if they are negatively related to each other, the covariance will be negative. Last, and very important, if they are unrelated, the covariance should be zero.

The formula for covariance can be written algebraically as:

$$\sigma_{AB} = \text{Cov}(R_A, R_B) = \text{Expected value of } [(R_A - \bar{R}_A) \times (R_B - \bar{R}_B)]$$

(continued)

where $\overline{R}_A$ and $\overline{R}_B$ are the expected returns for the two securities, and R_A and R_B are the actual returns. The ordering of the two variables is unimportant. That is, the covariance of A with B is equal to the covariance of B with A. This can be stated more formally as $\text{Cov}(R_A, R_B) = \text{Cov}(R_B, R_A)$ or $\sigma_{AB} = \sigma_{BA}$.

The covariance we calculated is -0.004875. A negative number like this implies that the return on one stock is likely to be above its average when the return on the other stock is below its average, and vice versa. However, the size of the number is difficult to interpret. Like the variance figure, the covariance is in squared deviation units. Until we can put it in perspective, we don't know what to make of it.

We solve the problem by computing the correlation.

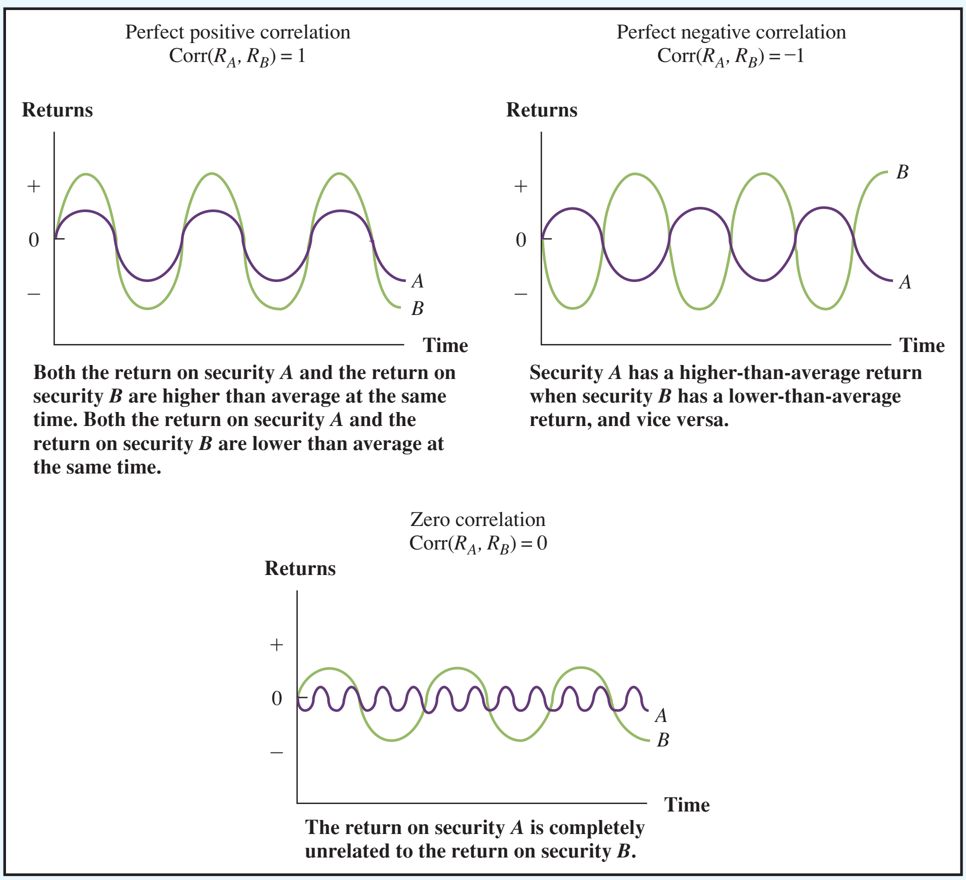
3. To calculate the correlation, divide the covariance by the standard deviations of both of the two securities. For our example, we have:

$$\rho_{AB} = \text{Corr}(R_A, R_B) = \frac{\text{Cov}(R_A, R_B)}{\sigma_A \times \sigma_B} = \frac{-0.004875}{0.2586 \times 0.1150} = -0.1639 \qquad \textbf{(10.2)}$$

where σ_A and σ_B are the standard deviations of Supertech and Slowpoke, respectively. Note that we represent the correlation between Supertech and Slowpoke either as $\text{Corr}(R_A, R_B)$ or ρ_{AB}. As with covariance, the ordering of the two variables is unimportant. That is, the correlation of A with B is equal to the correlation of B with A. More formally, $\text{Corr}(R_A, R_B) = \text{Corr}(R_B, R_A)$ or $\rho_{AB} = \rho_{BA}$.

Figure 10.1

Examples of Different Correlation Coefficients—Graphs Plotting the Separate Returns on Two Securities through Time



(continued)

Because the standard deviation is always positive, the sign of the correlation between two variables must be the same as that of the covariance between the two variables. If the correlation is positive, we say that the variables are *positively correlated*; if it is negative, we say that they are *negatively correlated*; and if it is zero, we say that they are *uncorrelated*. Furthermore, it can be proved that the correlation is always between $+1$ and -1. This is due to the standardizing procedure of dividing by the two standard deviations.

We can compare the correlation between different *pairs* of securities. For example, it turns out that the correlation between General Motors and Ford is much higher than the correlation between General Motors and IBM. Hence, we can state that the first pair of securities is more interrelated than the second pair.

Figure 10.1 shows the three benchmark cases for two assets, *A* and *B*. The figure shows two assets with return correlations of $+1$, -1, and 0. This implies perfect positive correlation, perfect negative correlation, and no correlation, respectively. The graphs in the figure plot the separate returns on the two securities through time.

10.3 The Return and Risk for Portfolios

Suppose an investor has estimates of the expected returns and standard deviations on individual securities and the correlations between securities. How does the investor choose the best combination or **portfolio** of securities to hold? Obviously, the investor would like a portfolio with a high expected return and a low standard deviation of return. It is therefore worthwhile to consider:

1. The relationship between the expected return on individual securities and the expected return on a portfolio made up of these securities.
2. The relationship between the standard deviations of individual securities, the correlations between these securities, and the standard deviation of a portfolio made up of these securities.

To analyze these two relationships, we will use the same example of Supertech and Slowpoke. The relevant calculations follow.

The Expected Return on a Portfolio

The formula for expected return on a portfolio is very simple:

The expected return on a portfolio is simply a weighted average of the expected returns on the individual securities.

Relevant Data from Example of Supertech and Slowpoke

Item	Symbol	Value
Expected return on Supertech	$\bar{R}_{Super}$	$0.175 = 17.5\%$
Expected return on Slowpoke	$\bar{R}_{Slow}$	$0.055 = 5.5\%$
Variance of Supertech	σ^2_{Super}	0.066875
Variance of Slowpoke	σ^2_{Slow}	0.013225
Standard deviation of Supertech	σ_{Super}	$0.2586 = 25.86\%$
Standard deviation of Slowpoke	σ_{Slow}	$0.1150 = 11.50\%$
Covariance between Supertech and Slowpoke	$\sigma_{Super, Slow}$	-0.004875
Correlation between Supertech and Slowpoke	$\rho_{Super, Slow}$	-0.1639

EXAMPLE 10.2

Portfolio Expected Returns Consider Supertech and Slowpoke. From our earlier calculations, we find that the expected returns on these two securities are 17.5 percent and 5.5 percent, respectively.

The expected return on a portfolio of these two securities alone can be written as:

$$\text{Expected return on portfolio} = X_{\text{Super}}\,(17.5\%) + X_{\text{Slow}}\,(5.5\%) = \overline{R}_P$$

where X_{Super} is the percentage of the portfolio in Supertech and X_{Slow} is the percentage of the portfolio in Slowpoke. If the investor with $100 invests $60 in Supertech and $40 in Slowpoke, the expected return on the portfolio can be written as:

$$\text{Expected return on portfolio} = 0.6 \times 17.5\% + 0.4 \times 5.5\% = 12.7\%$$

Algebraically, we can write:

$$\text{Expected return on portfolio} = X_A\overline{R}_A + X_B\overline{R}_B = \overline{R}_P \qquad \textbf{(10.3)}$$

where X_A and X_B are the proportions of the total portfolio in the assets A and B, respectively. (Because our investor can invest in only two securities, $X_A + X_B$ must equal 1 or 100 percent.) $\overline{R}_A$ and $\overline{R}_B$ are the expected returns on the two securities.

Now consider two stocks, each with an expected return of 10 percent. The expected return on a portfolio composed of these two stocks must be 10 percent, regardless of the proportions of the two stocks held. This result may seem obvious at this point, but it will become important later. The result implies that you do not reduce or *dissipate* your expected return by investing in a number of securities. Rather, the expected return on your portfolio is simply a weighted average of the expected returns on the individual assets in the portfolio.

Variance and Standard Deviation of a Portfolio

The Variance The formula for the variance of a portfolio composed of two securities, *A* and *B*, is:

The Variance of the Portfolio

$$\text{Var(portfolio)} = X_A^2\sigma_A^2 + 2X_AX_B\sigma_{A,B} + X_B^2\sigma_B^2$$

Note that there are three terms on the right side of the equation. The first term involves the variance of $A\,(\sigma_A^2)$, the second term involves the covariance between the two securities $(\sigma_{A,B})$, and the third term involves the variance of $B\,(\sigma_B^2)$. (As stated earlier in this chapter, $\sigma_{A,B} = \sigma_{B,A}$. That is, the ordering of the variables is not relevant when we are expressing the covariance between two securities.)

The formula indicates an important point. The variance of a portfolio depends on both the variances of the individual securities and the covariance between the two securities. The variance of a security measures the variability of an individual security's return. Covariance measures the relationship between the two securities. For given variances of the individual securities, a positive relationship or covariance between the two securities increases the variance of the entire portfolio. A negative relationship or covariance between the two securities decreases the variance of the entire portfolio. This important result seems to square with common sense. If one of your securities tends to go up when the other goes down, or vice versa, your two securities are offsetting each other. You are achieving what we call a *hedge* in finance, and the risk of your entire portfolio will be low. However, if both your securities rise and fall together, you are not hedging at all. Hence, the risk of your entire portfolio will be higher.

The variance formula for our two securities, Super and Slow, is:

$$\text{Var(portfolio)} = X^2_{\text{Super}}\sigma^2_{\text{Super}} + 2X_{\text{Super}}X_{\text{Slow}}\sigma_{\text{Super, Slow}} + X^2_{\text{Slow}}\sigma^2_{\text{Slow}} \quad \textbf{(10.4)}$$

Given our earlier assumption that an individual with \$100 invests \$60 in Supertech and \$40 in Slowpoke, $X_{\text{Super}} = 0.6$ and $X_{\text{Slow}} = 0.4$. Using this assumption and the relevant data from our previous calculations, the variance of the portfolio is:

$$0.023851 = 0.36 \times 0.066875 + 2 \times [0.6 \times 0.4 \times (-0.004875)] + 0.16 \times 0.013225 \quad \textbf{(10.4}'\textbf{)}$$

The Matrix Approach Alternatively, Equation 10.4 can be expressed in the following matrix format:

	Supertech	Slowpoke
Supertech	$X^2_{\text{Super}}\sigma^2_{\text{Super}}$ $0.024075 = 0.36 \times 0.066875$	$X_{\text{Super}}X_{\text{Slow}}\sigma_{\text{Super, Slow}}$ $-0.00117 = 0.6 \times 0.4 \times (-0.004875)$
Slowpoke	$X_{\text{Super}}X_{\text{Slow}}\sigma_{\text{Super, Slow}}$ $-0.00117 = 0.6 \times 0.4 \times (-0.004875)$	$X^2_{\text{Slow}}\sigma^2_{\text{Slow}}$ $0.002116 = 0.16 \times 0.013225$

There are four boxes in the matrix. We can add the terms in the boxes to obtain Equation 10.4, the variance of a portfolio composed of the two securities. The term in the upper left corner involves the variance of Supertech. The term in the lower right corner involves the variance of Slowpoke. The other two boxes contain the term involving the covariance. These two boxes are identical, indicating why the covariance term is multiplied by 2 in Equation 10.4.

At this point, students often find the box approach to be more confusing than Equation 10.4. However, the box approach is easily generalized to more than two securities, a task we perform later in this chapter.

Standard Deviation of a Portfolio Given Equation 10.4′, we can now determine the standard deviation of the portfolio's return. This is:

$$\sigma_P = \text{SD(portfolio)} = \sqrt{\text{Var(portfolio)}} = \sqrt{0.023851} \quad \textbf{(10.5)}$$
$$= 0.1544 = 15.44\%$$

The interpretation of the standard deviation of the portfolio is the same as the interpretation of the standard deviation of an individual security. The expected return on our portfolio is 12.7 percent. A return of −2.74 percent (=12.7% − 15.44%) is one standard deviation below the mean, and a return of 28.14 percent (=12.7% + 15.44%) is one standard deviation above the mean. If the return on the portfolio is normally distributed, a return between −2.74 percent and +28.14 percent occurs about 68 percent of the time.[3]

The Diversification Effect It is instructive to compare the standard deviation of the portfolio with the standard deviation of the individual securities. The weighted average of the standard deviations of the individual securities is:

$$\text{Weighted average of standard deviations} = X_{\text{Super}}\sigma_{\text{Super}} + X_{\text{Slow}}\sigma_{\text{Slow}} \quad \textbf{(10.6)}$$
$$0.2012 = 0.6 \times 0.2586 + 0.4 \times 0.115$$

[3] There are only four equally probable returns for Supertech and Slowpoke, so neither security possesses a normal distribution. Thus, probabilities would be slightly different in our example.

One of the most important results in this chapter concerns the difference between Equations 10.5 and 10.6. In our example, the standard deviation of the portfolio is *less* than a weighted average of the standard deviations of the individual securities.

We pointed out earlier that the expected return on the portfolio is a weighted average of the expected returns on the individual securities. Thus, we get a different type of result for the standard deviation of a portfolio than we do for the expected return on a portfolio.

It is generally argued that our result for the standard deviation of a portfolio is due to diversification. For example, Supertech and Slowpoke are slightly negatively correlated ($\rho = -0.1639$). Supertech's return is likely to be a little below average if Slowpoke's return is above average. Similarly, Supertech's return is likely to be a little above average if Slowpoke's return is below average. Thus, the standard deviation of a portfolio composed of the two securities is less than a weighted average of the standard deviations of the two securities.

Our example has negative correlation. Clearly, there will be less benefit from diversification if the two securities exhibit positive correlation. How high must the positive correlation be before all diversification benefits vanish?

To answer this question, let us rewrite Equation 10.4 in terms of correlation rather than covariance. The covariance can be rewritten as:[4]

$$\sigma_{\text{Super, Slow}} = \rho_{\text{Super, Slow}}\sigma_{\text{Super}}\sigma_{\text{Slow}} \tag{10.7}$$

This formula states that the covariance between any two securities is simply the correlation between the two securities multiplied by the standard deviations of each. In other words, covariance incorporates both (1) the correlation between the two assets and (2) the variability of each of the two securities as measured by standard deviation.

From our calculations earlier in this chapter we know that the correlation between the two securities is -0.1639. Given the variances used in Equation 10.4′, the standard deviations are 0.2586 and 0.115 for Supertech and Slowpoke, respectively. Thus, the variance of a portfolio can be expressed as follows:

Variance of the Portfolio's Return

$$= X^2_{\text{Super}}\sigma^2_{\text{Super}} + 2X_{\text{Super}}X_{\text{Slow}}\rho_{\text{Super, Slow}}\sigma_{\text{Super}}\sigma_{\text{Slow}} + X^2_{\text{Slow}}\sigma^2_{\text{Slow}} \tag{10.8}$$

$$\begin{aligned} 0.023851 = {} & 0.36 \times 0.066875 + 2 \times 0.6 \times 0.4 \times (-0.1639) \\ & \times 0.2586 \times 0.115 + 0.16 \times 0.013225 \end{aligned}$$

The middle term on the right side is now written in terms of correlation, ρ, not covariance.

Suppose $\rho_{\text{Super, Slow}} = 1$, the highest possible value for correlation. Assume all the other parameters in the example are the same. The variance of the portfolio is:

$$\begin{aligned} \text{Variance of the portfolio's return} = 0.040466 = {} & 0.36 \times 0.066875 + 2 \times (0.6 \times 0.4 \times 1 \times 0.2586 \\ & \times 0.115) + 0.16 \times 0.013225 \end{aligned}$$

The standard deviation is:

$$\text{Standard deviation of portfolio's return} = \sqrt{0.040466} = 0.2012 = 20.12\% \tag{10.9}$$

Note that Equations 10.9 and 10.6 are equal. That is, the standard deviation of a portfolio's return is equal to the weighted average of the standard deviations of the individual returns when $\rho = 1$. Inspection of Equation 10.8 indicates that the variance and hence the

[4] As with covariance, the ordering of the two securities is not relevant when we express the correlation between the two securities. That is, $\rho_{\text{Super,Slow}} = \rho_{\text{Slow,Super}}$.

Table 10.3

Standard Deviations for Standard & Poor's 500 Index and for Selected Stocks in the Index

Asset	Standard Deviation
S&P 500 Index	16.35%
Verizon	33.96
Ford Motor Co.	43.61
Walt Disney Co.	32.55
General Electric	25.18
IBM	35.96
McDonald's	28.61
Sears	44.06
Toys "R" Us Inc.	50.77
Amazon.com	69.19

As long as the correlations between pairs of securities are less than 1, the standard deviation of an index is less than the weighted average of the standard deviations of the individual securities within the index.

standard deviation of the portfolio must fall as the correlation drops below 1. This leads to the following result:

As long as $\rho < 1$, the standard deviation of a portfolio of two securities is *less* than the weighted average of the standard deviations of the individual securities.

In other words, the diversification effect applies as long as there is less than perfect correlation (as long as $\rho < 1$). Thus, our Supertech–Slowpoke example is a case of overkill. We illustrated diversification by an example with negative correlation. We could have illustrated diversification by an example with positive correlation—as long as it was not *perfect* positive correlation.

An Extension to Many Assets The preceding insight can be extended to the case of many assets. That is, as long as correlations between pairs of securities are less than 1, the standard deviation of a portfolio of many assets is less than the weighted average of the standard deviations of the individual securities.

Now consider Table 10.3, which shows the standard deviation of the Standard & Poor's 500 Index and the standard deviations of some of the individual securities listed in the index over a recent 10-year period. Note that all of the individual securities in the table have higher standard deviations than that of the index. In general, the standard deviations of most of the individual securities in an index will be above the standard deviation of the index itself, though a few of the securities could have lower standard deviations than that of the index.

10.4 The Efficient Set for Two Assets

Our results for expected returns and standard deviations are graphed in Figure 10.2. The figure shows a dot labeled Slowpoke and a dot labeled Supertech. Each dot represents both the expected return and the standard deviation for an individual security. As can be seen, Supertech has both a higher expected return and a higher standard deviation.

The box or "□" in the graph represents a portfolio with 60 percent invested in Supertech and 40 percent invested in Slowpoke. You will recall that we previously calculated both the expected return and the standard deviation for this portfolio.

The choice of 60 percent in Supertech and 40 percent in Slowpoke is just one of an infinite number of portfolios that can be created. The set of portfolios is sketched by the curved line in Figure 10.3.

Figure 10.2

Expected Returns and Standard Deviations for Supertech, Slowpoke, and a Portfolio Composed of 60 Percent in Supertech and 40 Percent in Slowpoke

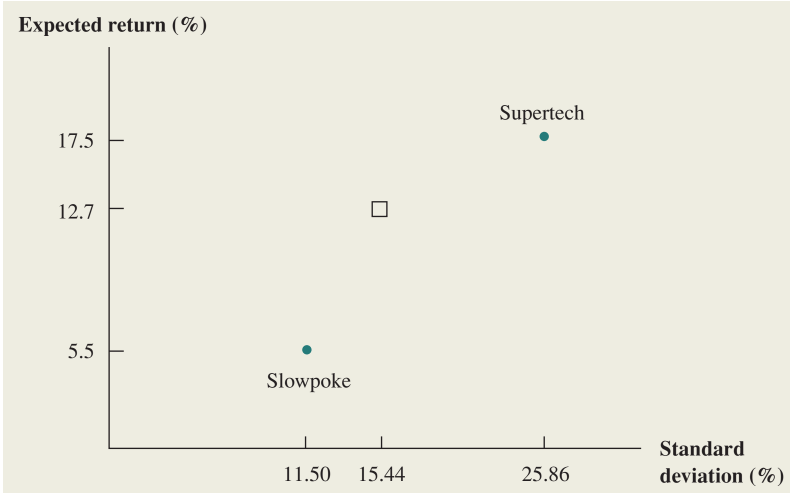
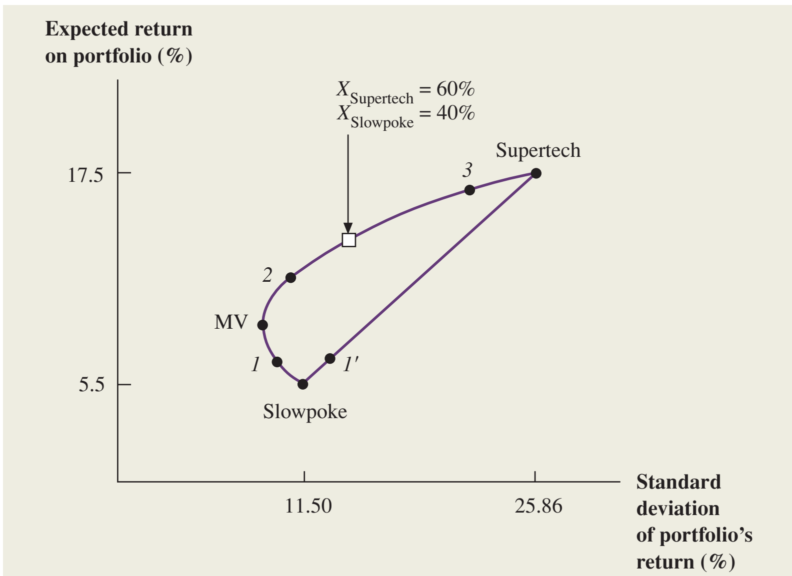


Figure 10.3

Set of Portfolios Composed of Holdings in Supertech and Slowpoke (correlation between the two securities is −0.1639)



Portfolio *1* is composed of 90 percent Slowpoke and 10 percent Supertech (ρ = −0.1639).
Portfolio *2* is composed of 50 percent Slowpoke and 50 percent Supertech (ρ = −0.1639).
Portfolio *3* is composed of 10 percent Slowpoke and 90 percent Supertech (ρ = −0.1639).
Portfolio *1′* is composed of 90 percent Slowpoke and 10 percent Supertech (ρ = 1).
Point MV denotes the minimum variance portfolio. This is the portfolio with the lowest possible variance. By definition, the same portfolio must also have the lowest possible standard deviation.

Consider portfolio *1*. This is a portfolio composed of 90 percent Slowpoke and 10 percent Supertech. Because it is weighted so heavily toward Slowpoke, it appears close to the Slowpoke point on the graph. Portfolio *2* is higher on the curve because it is composed of 50 percent Slowpoke and 50 percent Supertech. Portfolio *3* is close to the Supertech point on the graph because it is composed of 90 percent Supertech and 10 percent Slowpoke.

There are a few important points concerning this graph:

1. We argued that the diversification effect occurs whenever the correlation between the two securities is below 1. The correlation between Supertech and Slowpoke is -0.1639. The diversification effect can be illustrated by comparison with the straight line between the Supertech point and the Slowpoke point. The straight line represents points that would have been generated had the correlation coefficient between the two securities been 1. The diversification effect is illustrated in the figure because the curved line is always to the left of the straight line. Consider point *1′*. This represents a portfolio composed of 90 percent in Slowpoke and 10 percent in Supertech *if* the correlation between the two were exactly 1. We argue that there is no diversification effect if $\rho = 1$. However, the diversification effect applies to the curved line because point *1* has the same expected return as point *1′* but has a lower standard deviation. (Points *2′* and *3′* are omitted to reduce the clutter of Figure 10.3.)

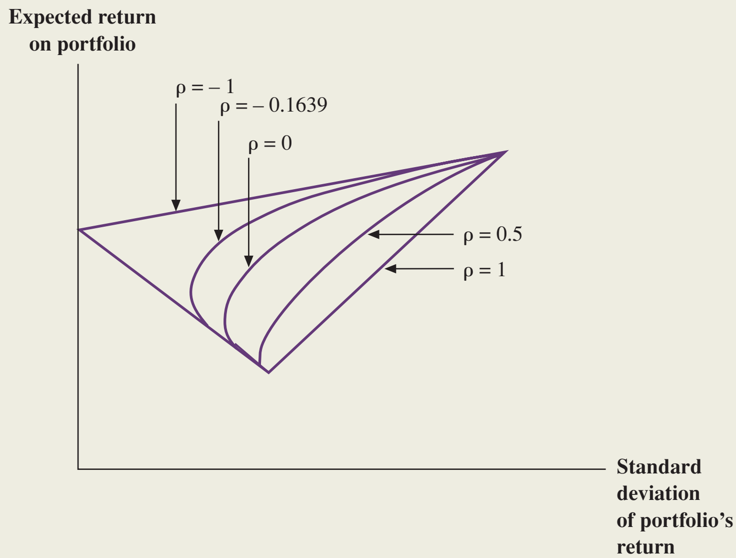
   Though the straight line and the curved line are both represented in Figure 10.3, they do not simultaneously exist in the same world. *Either* $\rho = -0.1639$ and the curve exists *or* $\rho = 1$ and the straight line exists. In other words, though an investor can choose between different points on the curve if $\rho = -0.1639$, she cannot choose between points on the curve and points on the straight line.
2. The point MV represents the minimum variance portfolio. This is the portfolio with the lowest possible variance. By definition, this portfolio must also have the lowest possible standard deviation. (The term *minimum variance portfolio* is standard in the literature, and we will use that term. Perhaps minimum standard deviation would actually be better because standard deviation, not variance, is measured on the horizontal axis of Figure 10.3.)
3. An individual contemplating an investment in a portfolio of Slowpoke and Supertech faces an **opportunity set** or **feasible set** represented by the curved line in Figure 10.3. That is, he can achieve any point on the curve by selecting the appropriate mix between the two securities. He cannot achieve any point above the curve because he cannot increase the return on the individual securities, decrease the standard deviations of the securities, or decrease the correlation between the two securities. Neither can he achieve points below the curve because he cannot lower the returns on the individual securities, increase the standard deviations of the securities, or increase the correlation. (Of course, he would not want to achieve points below the curve, even if he were able to do so.)

   Were he relatively tolerant of risk, he might choose portfolio *3*. (In fact, he could even choose the end point by investing all his money in Supertech.) An investor with less tolerance for risk might choose portfolio *2*. An investor wanting as little risk as possible would choose MV, the portfolio with minimum variance or minimum standard deviation.
4. Note that the curve is backward bending between the Slowpoke point and MV. This indicates that, for a portion of the feasible set, standard deviation actually decreases as we increase expected return. Students frequently ask, "How can an increase in the proportion of the risky security, Supertech, lead to a reduction in the risk of the portfolio?"

   This surprising finding is due to the diversification effect. The returns on the two securities are negatively correlated with each other. One security tends to go up when the other goes down and vice versa. Thus, an addition of a small amount of Supertech acts as a hedge to a portfolio composed only of Slowpoke. The risk of the portfolio is reduced, implying backward bending. Actually, backward bending always occurs if $\rho \leq 0$. It may or may not occur when $\rho > 0$. Of course, the curve bends backward

Figure 10.4

Opportunity Sets Composed of Holdings in Supertech and Slowpoke



Each curve represents a different correlation. The lower the correlation, the more bend in the curve.

only for a portion of its length. As we continue to increase the percentage of Supertech in the portfolio, the high standard deviation of this security eventually causes the standard deviation of the entire portfolio to rise.

5. No investor would want to hold a portfolio with an expected return below that of the minimum variance portfolio. For example, no investor would choose portfolio *1*. This portfolio has less expected return but more standard deviation than the minimum variance portfolio has. We say that portfolios such as portfolio *1* are *dominated* by the minimum variance portfolio. Though the entire curve from Slowpoke to Supertech is called the *feasible set*, investors consider only the curve from MV to Supertech. Hence the curve from MV to Supertech is called the **efficient set** or the **efficient frontier**.

Figure 10.3 represents the opportunity set where $\rho = -0.1639$. It is worthwhile to examine Figure 10.4, which shows different curves for different correlations. As can be seen, the lower the correlation, the more bend there is in the curve. This indicates that the diversification effect rises as ρ declines. The greatest bend occurs in the limiting case where $\rho = -1$. This is perfect negative correlation. While this extreme case where $\rho = -1$ seems to fascinate students, it has little practical importance. Most pairs of securities exhibit positive correlation. Strong negative correlations, let alone perfect negative correlation, are unlikely occurrences indeed.[5]

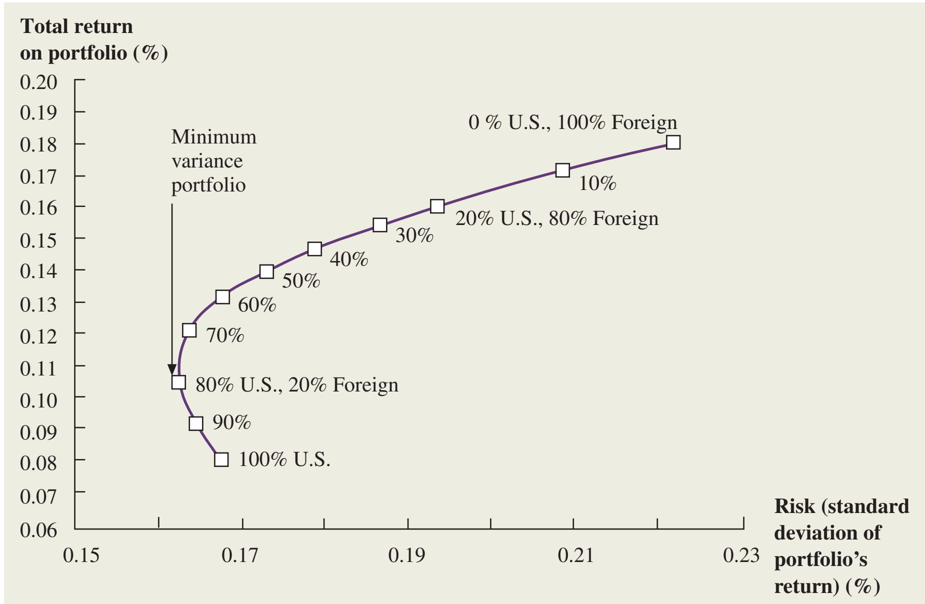
Note that there is only one correlation between a pair of securities. We stated earlier that the correlation between Slowpoke and Supertech is -0.1639. Thus, the curve in Figure 10.4 representing this correlation is the correct one, and the other curves should be viewed as merely hypothetical.

The graphs we examined are not mere intellectual curiosities. Rather, efficient sets can easily be calculated in the real world. As mentioned earlier, data on returns, standard deviations, and correlations are generally taken from past observations, though subjective notions can be used to determine the values of these parameters as well. Once the parameters

[5] A major exception occurs with derivative securities. For example, the correlation between a stock and a put on the stock is generally strongly negative. Puts will be treated later in the text.

Figure 10.5

Return/Risk Trade-off for World Stocks: Portfolio of U.S. and Foreign Stocks



have been determined, any one of a whole host of software packages can be purchased to generate an efficient set. However, the choice of the preferred portfolio within the efficient set is up to you. As with other important decisions like what job to choose, what house or car to buy, and how much time to allocate to this course, there is no computer program to choose the preferred portfolio.

An efficient set can be generated where the two individual assets are portfolios themselves. For example, the two assets in Figure 10.5 are a diversified portfolio of American stocks and a diversified portfolio of foreign stocks. Expected returns, standard deviations, and the correlation coefficient were calculated over the recent past. No subjectivity entered the analysis. The U.S. stock portfolio with a standard deviation of about 0.173 is less risky than the foreign stock portfolio, which has a standard deviation of about 0.222. However, combining a small percentage of the foreign stock portfolio with the U.S. portfolio actually reduces risk, as can be seen by the backward-bending nature of the curve. In other words, the diversification benefits from combining two different portfolios more than offset the introduction of a riskier set of stocks into our holdings. The minimum variance portfolio occurs with about 80 percent of our funds in American stocks and about 20 percent in foreign stocks. Addition of foreign securities beyond this point increases the risk of the entire portfolio.

The backward-bending curve in Figure 10.5 is important information that has not bypassed American money managers. In recent years, pension fund and mutual fund managers in the United States have sought investment opportunities overseas. Another point worth pondering concerns the potential pitfalls of using only past data to estimate future returns. The stock markets of many foreign countries have had phenomenal growth in the past 25 years. Thus, a graph like Figure 10.5 makes a large investment in these foreign markets seem attractive. However, because abnormally high returns cannot be sustained forever, some subjectivity must be used in forecasting future expected returns.

10.5 The Efficient Set for Many Securities

The previous discussion concerned two securities. We found that a simple curve sketched out all the possible portfolios. Because investors generally hold more than two securities, we should look at the same graph when more than two securities are held. The shaded area in

Figure 10.6

The Feasible Set of Portfolios Constructed from Many Securities

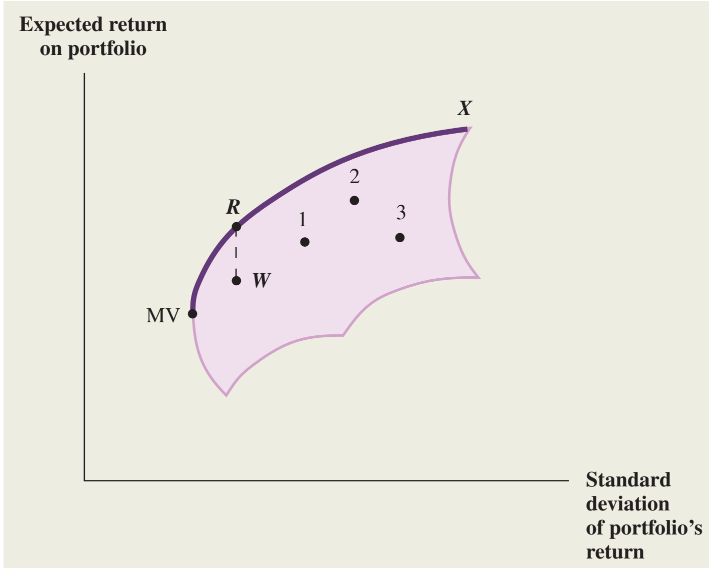


Figure 10.6 represents the opportunity set or feasible set when many securities are considered. The shaded area represents all the possible combinations of expected return and standard deviation for a portfolio. For example, in a universe of 100 securities, point 1 might represent a portfolio of, say, 40 securities. Point 2 might represent a portfolio of 80 securities. Point 3 might represent a different set of 80 securities, or the same 80 securities held in different proportions, or something else. Obviously, the combinations are virtually endless. However, note that all possible combinations fit into a confined region. No security or combination of securities can fall outside the shaded region. That is, no one can choose a portfolio with an expected return above that given by the shaded region. Furthermore, no one can choose a portfolio with a standard deviation below that given in the shaded area. Perhaps more surprisingly, no one can choose an expected return below that given in the curve. In other words, the capital markets actually prevent a self-destructive person from taking on a guaranteed loss.[6]

So far, Figure 10.6 is different from the earlier graphs. When only two securities are involved, all the combinations lie on a single curve. Conversely, with many securities the combinations cover an entire area. However, notice that an individual will want to be somewhere on the upper edge between MV and *X*. The upper edge, which we indicate in Figure 10.6 by a thick curve, is called the *efficient set*. Any point below the efficient set would receive less expected return and the same standard deviation as a point on the efficient set. For example, consider *R* on the efficient set and *W* directly below it. If *W* contains the risk level you desire, you should choose *R* instead to receive a higher expected return.

In the final analysis, Figure 10.6 is quite similar to Figure 10.3. The efficient set in Figure 10.3 runs from MV to Supertech. It contains various combinations of the securities Supertech and Slowpoke. The efficient set in Figure 10.6 runs from MV to *X*. It contains various combinations of many securities. The fact that a whole shaded area appears in Figure 10.6 but not in Figure 10.3 is just not an important difference; no investor would choose any point below the efficient set in Figure 10.6 anyway.

We mentioned before that an efficient set for two securities can be traced out easily in the real world. The task becomes more difficult when additional securities are included

[6]Of course, someone dead set on parting with his money can do so. For example, he can trade frequently without purpose, so that commissions more than offset the positive expected returns on the portfolio.

Table 10.4

Matrix Used to Calculate the Variance of a Portfolio

Stock	1	2	3	...	N
1	$X_1^2\sigma_1^2$	$X_1X_2\text{Cov}(R_1,R_2)$	$X_1X_3\text{Cov}(R_1,R_3)$		$X_1X_N\text{Cov}(R_1,R_N)$
2	$X_2X_1\text{Cov}(R_2,R_1)$	$X_2^2\sigma_2^2$	$X_2X_3\text{Cov}(R_2,R_3)$		$X_2X_N\text{Cov}(R_2,R_N)$
3	$X_3X_1\text{Cov}(R_3,R_1)$	$X_3X_2\text{Cov}(R_3,R_2)$	$X_3^2\sigma_3^2$		$X_3X_N\text{Cov}(R_3,R_N)$
.					
.					
.					
N	$X_NX_1\text{Cov}(R_N,R_1)$	$X_NX_2\text{Cov}(R_N,R_2)$	$X_NX_3\text{Cov}(R_N,R_3)$		$X_N^2\sigma_N^2$

The variance of the portfolio is the sum of the terms in all the boxes.

σ_i is the standard deviation of stock *i*.

$\text{Cov}(R_i, R_j)$ is the covariance between stock *i* and stock *j*.

Terms involving the standard deviation of a single security appear on the diagonal. Terms involving covariance between two securities appear off the diagonal.

because the number of observations grows. For example, using subjective analysis to estimate expected returns and standard deviations for, say, 100 or 500 securities may very well become overwhelming, and the difficulties with correlations may be greater still. There are almost 5,000 correlations between pairs of securities from a universe of 100 securities.

Though much of the mathematics of efficient set computation had been derived in the 1950s,[7] the high cost of computer time restricted application of the principles. In recent years this cost has been drastically reduced. A number of software packages allow the calculation of an efficient set for portfolios of moderate size. By all accounts these packages sell quite briskly, so our discussion would appear to be important in practice.

Variance and Standard Deviation in a Portfolio of Many Assets

We earlier calculated the formulas for variance and standard deviation in the two-asset case. Because we considered a portfolio of many assets in Figure 10.6, it is worthwhile to calculate the formulas for variance and standard deviation in the many-asset case. The formula for the variance of a portfolio of many assets can be viewed as an extension of the formula for the variance of two assets.

To develop the formula, we employ the same type of matrix that we used in the two-asset case. This matrix is displayed in Table 10.4. Assuming that there are N assets, we write the numbers 1 through N on the horizontal axis and 1 through N on the vertical axis. This creates a matrix of $N \times N = N^2$ boxes. The variance of the portfolio is the sum of the terms in all the boxes.

Consider, for example, the box in the second row and the third column. The term in the box is X_2X_3 Cov(R_2,R_3). X_2 and X_3 are the percentages of the entire portfolio that are invested in the second asset and the third asset, respectively. For example, if an individual with a portfolio of \$1,000 invests \$100 in the second asset, $X_2 = 10\%$ (=\$100/\$1,000). Cov(R_3,R_2) is the covariance between the returns on the third asset and the returns on the second asset. Next, note the box in the third row and the second column. The term in this box is X_3X_2 Cov(R_3,R_2). Because Cov(R_3,R_2) = Cov(R_2,R_3), both boxes have the same value. The second security and the third security make up one pair of stocks. In fact, every pair of stocks appears twice in the table: once in the lower left side and once in the upper right side.

Now consider boxes on the diagonal. For example, the term in the first box on the diagonal is $X_1^2\sigma_1^2$. Here, σ_1^2 is the variance of the return on the first security.

[7]The classic treatise is Harry Markowitz, *Portfolio Selection* (New York: John Wiley & Sons, 1959). Markowitz won the Nobel Prize in economics in 1990 for his work on modern portfolio theory.

Table 10.5

Number of Variance and Covariance Terms as a Function of the Number of Stocks in the Portfolio

Number of Stocks in Portfolio	Total Number of Terms	Number of Variance Terms (number of terms on diagonal)	Number of Covariance Terms (number of terms off diagonal)
1	1	1	0
2	4	2	2
3	9	3	6
10	100	10	90
100	10,000	100	9,900
.	.	.	.
.	.	.	.
.	.	.	.
N	N^2	N	$N^2 - N$

In a large portfolio, the number of terms involving covariance between two securities is much greater than the number of terms involving variance of a single security.

Thus, the diagonal terms in the matrix contain the variances of the different stocks. The off-diagonal terms contain the covariances. Table 10.5 relates the numbers of diagonal and off-diagonal elements to the size of the matrix. The number of diagonal terms (number of variance terms) is always the same as the number of stocks in the portfolio. The number of off-diagonal terms (number of covariance terms) rises much faster than the number of diagonal terms. For example, a portfolio of 100 stocks has 9,900 covariance terms. Because the variance of a portfolio's return is the sum of all the boxes, we have the following:

The variance of the return on a portfolio with many securities is more dependent on the covariances between the individual securities than on the variances of the individual securities.

To give a recent example of the impact of diversification, the Dow Jones Industrial Average (DJIA), which contains 30 large, well-known U.S. stocks, was about flat in 2005, meaning no gain or loss. As we saw in our previous chapter, this performance represents a fairly bad year for a portfolio of large-cap stocks. The biggest individual gainers for the year were Hewlett Packard (up 37 percent). Boeing (up 36 percent), and Altria Group (up 22 percent). However, offsetting these nice gains were General Motors (down 52 percent), Verizon Communications (down 26 percent), and IBM (down 17 percent). So, there were big winners and big losers, and they more or less offset in this particular year.

10.6 Diversification: An Example

The preceding point can be illustrated by altering the matrix in Table 10.4 slightly. Suppose we make the following three assumptions:

1. All securities possess the same variance, which we write as $\overline{\text{var}}$. In other words, $\sigma_i^2 = \overline{\text{var}}$ for every security.
2. All covariances in Table 10.4 are the same. We represent this uniform covariance as $\overline{\text{cov}}$. In other words. $\text{Cov}(R_i, R_j) = \overline{\text{cov}}$ for every pair of securities. It can easily be shown that $\overline{\text{var}} > \overline{\text{cov}}$.
3. All securities are equally weighted in the portfolio. Because there are N assets, the weight of each asset in the portfolio is $1/N$. In other words, $X_i = 1/N$ for each security i.

Table 10.6 **Matrix Used to Calculate the Variance of a Portfolio When (a) All Securities Possess the Same Variance, Which We Represent as $\overline{\text{var}}$; (b) All Pairs of Securities Possess the Same Covariance, Which We Represent as $\overline{\text{cov}}$; (c) All Securities Are Held in the Same Proportion, Which Is 1/*N***

Stock	*1*	*2*	*3*	...	*N*
1	$(1/N^2)\ \overline{\text{var}}$	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{cov}}$		$(1/N^2)\ \overline{\text{cov}}$
2	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{var}}$	$(1/N^2)\ \overline{\text{cov}}$		$(1/N^2)\ \overline{\text{cov}}$
3	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{var}}$		$(1/N^2)\ \overline{\text{cov}}$
.					
.					
.					
N	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{cov}}$	$(1/N^2)\ \overline{\text{cov}}$		$(1/N^2)\ \overline{\text{var}}$

Table 10.6 is the matrix of variances and covariances under these three simplifying assumptions. Note that all of the diagonal terms are identical. Similarly, all of the off-diagonal terms are identical. As with Table 10.4, the variance of the portfolio is the sum of the terms in the boxes in Table 10.6. We know that there are N diagonal terms involving variance. Similarly, there are $N \times (N - 1)$ off-diagonal terms involving covariance. Summing across all the boxes in Table 10.6, we can express the variance of the portfolio as:

$$\text{Variance of portfolio} = \underbrace{N}_{\substack{\text{Number of}\\ \text{diagonal}\\ \text{terms}}} \times \underbrace{\left(\frac{1}{N^2}\right)\overline{\text{var}}}_{\substack{\text{Each}\\ \text{diagonal}\\ \text{term}}} + \underbrace{N(N-1)}_{\substack{\text{Number of}\\ \text{off-diagonal}\\ \text{terms}}} \times \underbrace{\left(\frac{1}{N^2}\right)\overline{\text{cov}}}_{\substack{\text{Each}\\ \text{off-diagonal}\\ \text{term}}} \quad \textbf{(10.10)}$$

$$= \left(\frac{1}{N}\right)\overline{\text{var}} + \left(\frac{N^2 - N}{N^2}\right)\overline{\text{cov}}$$

$$= \left(\frac{1}{N}\right)\overline{\text{var}} + \left(1 - \frac{1}{N}\right)\overline{\text{cov}}$$

Equation 10.10 expresses the variance of our special portfolio as a weighted sum of the average security variance and the average covariance.[8]

Now, let's increase the number of securities in the portfolio without limit. The variance of the portfolio becomes:

$$\text{Variance of portfolio (when } N \to \infty) = \overline{\text{cov}} \quad \textbf{(10.11)}$$

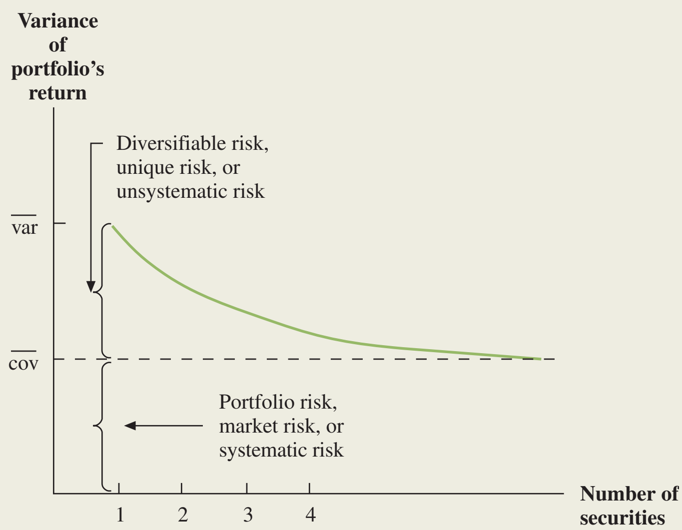
This occurs because (1) the weight on the variance term, $1/N$, goes to 0 as N goes to infinity, and (2) the weight on the covariance term, $1 - 1/N$, goes to 1 as N goes to infinity.

Equation 10.11 provides an interesting and important result. In our special portfolio, the variances of the individual securities completely vanish as the number of securities becomes large. However, the covariance terms remain. In fact, the variance of the portfolio becomes the average covariance, $\overline{\text{cov}}$. We often hear that we should diversify. In other words, we should not put all our eggs in one basket. The effect of diversification on the risk of a portfolio can be illustrated in this example. The variances of the individual securities are diversified away, but the covariance terms cannot be diversified away.

[8] Equation 10.10 is actually a weighted *average* of the variance and covariance terms because the weights, $1/N$ and $1 - 1/N$, sum to 1.

Figure 10.7

Relationship between the Variance of a Portfolio's Return and the Number of Securities in the Portfolio*



*** This graph assumes**

a. All securities have constant variance, var.

b. All securities have constant covariance, cov.

c. All securities are equally weighted in the portfolio.

The variance of a portfolio drops as more securities are added to the portfolio. However, it does not drop to zero. Rather, cov serves as the floor.

The fact that part, but not all, of our risk can be diversified away should be explored. Consider Mr. Smith, who brings $1,000 to the roulette table at a casino. It would be very risky if he put all his money on one spin of the wheel. For example, imagine that he put the full $1,000 on red at the table. If the wheel showed red, he would get $2,000; but if the wheel showed black, he would lose everything. Suppose instead he divided his money over 1,000 different spins by betting $1 at a time on red. Probability theory tells us that he could count on winning about 50 percent of the time. This means he could count on pretty nearly getting all his original $1,000 back.[9] In other words, risk is essentially eliminated with 1,000 different spins.

Now, let's contrast this with our stock market example, which we illustrate in Figure 10.7. The variance of the portfolio with only one security is, of course, $\overline{\text{var}}$ because the variance of a portfolio with one security is the variance of the security. The variance of the portfolio drops as more securities are added, which is evidence of the diversification effect. However, unlike Mr. Smith's roulette example, the portfolio's variance can never drop to zero. Rather it reaches a floor of $\overline{\text{cov}}$, which is the covariance of each pair of securities.[10]

Because the variance of the portfolio asymptotically approaches $\overline{\text{cov}}$, each additional security continues to reduce risk. Thus, if there were neither commissions nor other transactions costs, it could be argued that we can never achieve too much diversification. However, there is a cost to diversification in the real world. Commissions per dollar invested fall as we make larger purchases in a single stock. Unfortunately, we must buy fewer shares of each security when buying more and more different securities. Comparing the costs

[9]This example ignores the casino's cut.

[10]Though it is harder to show, this risk reduction effect also applies to the general case where variances and covariances are *not* equal.

and benefits of diversification, Meir Statman argues that a portfolio of about 30 stocks is needed to achieve optimal diversification.[11]

We mentioned earlier that $\overline{\text{var}}$ must be greater than $\overline{\text{cov}}$. Thus, the variance of a security's return can be broken down in the following way:

$$\begin{matrix}\text{Total risk of} \\ \text{individual security} \\ (\overline{\text{var}})\end{matrix} = \begin{matrix}\text{Portfolio risk} \\ (\overline{\text{cov}})\end{matrix} + \begin{matrix}\text{Unsystematic or} \\ \text{diversifiable risk} \\ (\overline{\text{var}} - \overline{\text{cov}})\end{matrix}$$

Total risk, which is $\overline{\text{var}}$ in our example, is the risk we bear by holding onto one security only. *Portfolio risk* is the risk we still bear after achieving full diversification, which is $\overline{\text{cov}}$ in our example. Portfolio risk is often called **systematic** or **market risk** as well. **Diversifiable**, **unique**, or **unsystematic risk** is the risk that can be diversified away in a large portfolio, which must be ($\overline{\text{var}} - \overline{\text{cov}}$) by definition.

To an individual who selects a diversified portfolio, the total risk of an individual security is not important. When considering adding a security to a diversified portfolio, the individual cares about only that portion of the risk of a security that cannot be diversified away. This risk can alternatively be viewed as the *contribution* of a security to the risk of an entire portfolio. We will talk later about the case where securities make different contributions to the risk of the entire portfolio.

Risk and the Sensible Investor

Having gone to all this trouble to show that unsystematic risk disappears in a well-diversified portfolio, how do we know that investors even want such portfolios? What if they like risk and don't want it to disappear?

We must admit that, theoretically at least, this is possible, but we will argue that it does not describe what we think of as the typical investor. Our typical investor is **risk-averse**. Risk-averse behavior can be defined in many ways, but we prefer the following example: A fair gamble is one with zero expected return; a risk-averse investor would prefer to avoid fair gambles.

Why do investors choose well-diversified portfolios? Our answer is that they are risk-averse, and risk-averse people avoid unnecessary risk, such as the unsystematic risk on a stock. If you do not think this is much of an answer, consider whether you would take on such a risk. For example, suppose you had worked all summer and had saved \$5,000, which you intended to use for your college expenses. Now, suppose someone came up to you and offered to flip a coin for the money: heads, you would double your money, and tails, you would lose it all.

Would you take such a bet? Perhaps you would, but most people would not. Leaving aside any moral question that might surround gambling and recognizing that some people would take such a bet, it's our view that the average investor would not.

To induce the typical risk-averse investor to take a fair gamble, you must sweeten the pot. For example, you might need to raise the odds of winning from 50–50 to 70–30 or higher. The risk-averse investor can be induced to take fair gambles only if they are sweetened so that they become unfair to the investor's advantage.

10.7 Riskless Borrowing and Lending

Figure 10.6 assumes that all the securities in the efficient set are risky. Alternatively, an investor could combine a risky investment with an investment in a riskless or *risk-free* security, such as an investment in U.S. Treasury bills. This is illustrated in the following example.

[11] Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* (September 1987).

EXAMPLE 10.3

Riskless Lending and Portfolio Risk Ms. Bagwell is considering investing in the common stock of Merville Enterprises. In addition, Ms. Bagwell will either borrow or lend at the risk-free rate. The relevant parameters are these:

	Common Stock of Merville	Risk-Free Asset
Expected return	14%	10%
Standard deviation	0.20	0

Suppose Ms. Bagwell chooses to invest a total of $1,000, $350 of which is to be invested in Merville Enterprises and $650 placed in the risk-free asset. The expected return on her total investment is simply a weighted average of the two returns:

$$\begin{array}{c}\text{Expected return on portfolio}\\ \text{composed of one riskless}\\ \text{and one risky asset}\end{array} = 0.114 = (0.35 \times 0.14) + (0.65 \times 0.10) \qquad \textbf{(10.12)}$$

Because the expected return on the portfolio is a weighted average of the expected return on the risky asset (Merville Enterprises) and the risk-free return, the calculation is analogous to the way we treated two risky assets. In other words, Equation 10.3 applies here.

Using Equation 10.4, the formula for the variance of the portfolio can be written as:

$$X^2_{\text{Merville}}\sigma^2_{\text{Merville}} + 2X_{\text{Merville}}X_{\text{Risk-free}}\sigma_{\text{Merville, Risk-free}} + X^2_{\text{Risk-free}}\sigma^2_{\text{Risk-free}}$$

However, by definition, the risk-free asset has no variability. Thus both $\sigma_{\text{Merville, Risk-free}}$ and $\sigma^2_{\text{Risk-free}}$ are equal to zero, reducing the above expression to:

$$\begin{array}{c}\text{Variance of portfolio composed}\\ \text{of one riskless and one risky asset}\end{array} = X^2_{\text{Merville}}\sigma^2_{\text{Merville}} \qquad \textbf{(10.13)}$$

$$= (0.35)^2 \times (0.20)^2$$

$$= 0.0049$$

The standard deviation of the portfolio is:

$$\begin{array}{c}\text{Standard deviation of portfolio composed}\\ \text{of one riskless and one risky asset}\end{array} = X_{\text{Merville}}\sigma_{\text{Merville}} \qquad \textbf{(10.14)}$$

$$= 0.35 \times 0.20$$

$$= 0.07$$

The relationship between risk and expected return for one risky and one riskless asset can be seen in Figure 10.8. Ms. Bagwell's split of 35–65 percent between the two assets is represented on a *straight* line between the risk-free rate and a pure investment in Merville Enterprises. Note that, unlike the case of two risky assets, the opportunity set is straight, not curved.

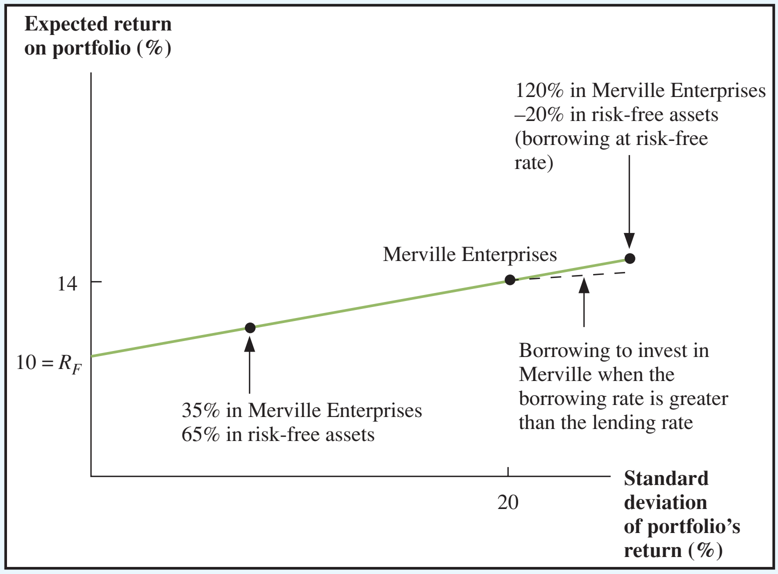
Suppose that, alternatively, Ms. Bagwell borrows $200 at the risk-free rate. Combining this with her original sum of $1,000, she invests a total of $1,200 in Merville. Her expected return would be:

$$\begin{array}{c}\text{Expected return on portfolio}\\ \text{formed by borrowing}\\ \text{to invest in risky asset}\end{array} = 14.8\% = 1.20 \times 0.14 + (-0.2 \times 0.10)$$

Here, she invests 120 percent of her original investment of $1,000 by borrowing 20 percent of her original investment. Note that the return of 14.8 percent is greater than the 14 percent expected return on Merville Enterprises. This occurs because she is borrowing at 10 percent to invest in a security with an expected return greater than 10 percent.

(*continued*)

Figure 10.8 **Relationship between Expected Return and Risk for a Portfolio of One Risky Asset and One Riskless Asset**



The standard deviation is:

$$\begin{array}{c}\text{Standard deviation of portfolio formed} \\ \text{by borrowing to invest in risky asset}\end{array} = 0.24 = 1.20 \times 0.2$$

The standard deviation of 0.24 is greater than 0.20, the standard deviation of the Merville investment, because borrowing increases the variability of the investment. This investment also appears in Figure 10.8.

So far, we have assumed that Ms. Bagwell is able to borrow at the same rate at which she can lend.[12] Now let us consider the case where the borrowing rate is above the lending rate. The dotted line in Figure 10.8 illustrates the opportunity set for borrowing opportunities in this case. The dotted line is below the solid line because a higher borrowing rate lowers the expected return on the investment.

The Optimal Portfolio

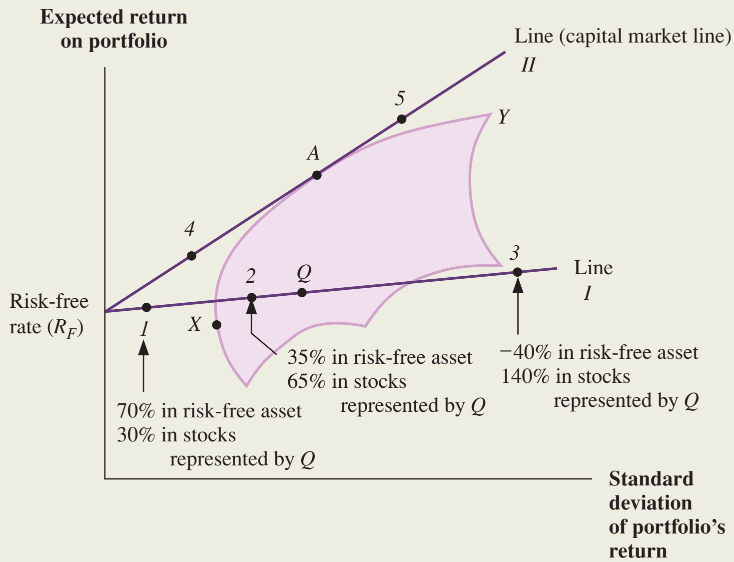
The previous section concerned a portfolio formed between one riskless asset and one risky asset. In reality, an investor is likely to combine an investment in the riskless asset with a *portfolio* of risky assets. This is illustrated in Figure 10.9.

Consider point Q, representing a portfolio of securities. Point Q is in the interior of the feasible set of risky securities. Let us assume the point represents a portfolio of 30 percent in AT&T, 45 percent in General Motors (GM), and 25 percent in IBM. Individuals combining investments in Q with investments in the riskless asset would achieve points along the straight line from R_F to Q. We refer to this as line *I*. For example, point *1* on the line represents a portfolio of 70 percent in the riskless asset and 30 percent in stocks represented by Q. An investor with \$100 choosing point *1* as his portfolio would put \$70 in

[12] Surprisingly, this appears to be a decent approximation because many investors can borrow from a stockbroker (called *going on margin*) when purchasing stocks. The borrowing rate here is very near the riskless rate of interest, particularly for large investors. More will be said about this in a later chapter.

Figure 10.9

Relationship between Expected Return and Standard Deviation for an Investment in a Combination of Risky Securities and the Riskless Asset



Portfolio *Q* is composed of 30 percent AT&T, 45 percent GM, and 25 percent IBM.

the risk-free asset and \$30 in *Q*. This can be restated as \$70 in the riskless asset, \$9 (=0.3 × \$30) in AT&T, \$13.50 (= 0.45 × \$30) in GM, and \$7.50 (= 0.25 × \$30) in IBM. Point *2* also represents a portfolio of the risk-free asset and *Q*, with more (65%) being invested in *Q*.

Point *3* is obtained by borrowing to invest in *Q*. For example, an investor with \$100 of her own would borrow \$40 from the bank or broker to invest \$140 in *Q*. This can be stated as borrowing \$40 and contributing \$100 of her money to invest \$42 (= 0.3 × \$140) in AT&T, \$63 (= 0.45 × \$140) in GM, and \$35 (= 0.25 × \$140) in IBM.

These investments can be summarized as follows:

	Point Q	Point *1* (Lending \$70)	Point *3* (Borrowing \$40)
AT&T	\$ 30	\$ 9	\$ 42
GM	45	13.50	63
IBM	25	7.50	35
Risk-free	0	70.00	−40
Total investment	\$100	\$100	\$100

Though any investor can obtain any point on line *I*, no point on the line is optimal. To see this, consider line *II*, a line running from R_F through *A*. Point *A* represents a portfolio of risky securities. Line *II* represents portfolios formed by combinations of the risk-free asset and the securities in *A*. Points between R_F and *A* are portfolios in which some money is invested in the riskless asset and the rest is placed in *A*. Points past *A* are achieved by borrowing at the riskless rate to buy more of *A* than we could with our original funds alone.

As drawn, line *II* is tangent to the efficient set of risky securities. Whatever point an individual can obtain on line *I*, he can obtain a point with the same standard deviation and a higher expected return on line *II*. In fact, because line *II* is tangent to the efficient set of risky assets, it provides the investor with the best possible opportunities. In other words, line *II* can be viewed as the efficient set of *all* assets, both risky and riskless. An investor with a fair degree of risk aversion might choose a point between R_F and A, perhaps point *4*. An individual with less risk aversion might choose a point closer to A or even beyond A. For example, point *5* corresponds to an individual borrowing money to increase investment in A.

The graph illustrates an important point. With riskless borrowing and lending, the portfolio of *risky* assets held by any investor would always be point A. Regardless of the investor's tolerance for risk, she would never choose any other point on the efficient set of risky assets (represented by curve XAY) nor any point in the interior of the feasible region. Rather, she would combine the securities of A with the riskless assets if she had high aversion to risk. She would borrow the riskless asset to invest more funds in A had she low aversion to risk.

This result establishes what financial economists call the **separation principle**. That is, the investor's investment decision consists of two separate steps:

1. After estimating (*a*) the expected returns and variances of individual securities, and (*b*) the covariances between pairs of securities, the investor calculates the efficient set of risky assets, represented by curve XAY in Figure 10.9. He then determines point A, the tangency between the risk-free rate and the efficient set of risky assets (curve XAY). Point A represents the portfolio of risky assets that the investor will hold. This point is determined solely from his estimates of returns, variances, and covariances. No personal characteristics, such as degree of risk aversion, are needed in this step.
2. The investor must now determine how he will combine point A, his portfolio of risky assets, with the riskless asset. He might invest some of his funds in the riskless asset and some in portfolio A. He would end up at a point on the line between R_F and A in this case. Alternatively, he might borrow at the risk-free rate and contribute some of his own funds as well, investing the sum in portfolio A. He would end up at a point on line II beyond A. His position in the riskless asset—that is, his choice of where on the line he wants to be—is determined by his internal characteristics, such as his ability to tolerate risk.

10.8 Market Equilibrium

Definition of the Market Equilibrium Portfolio

The preceding analysis concerns one investor. His estimates of the expected returns and variances for individual securities and the covariances between pairs of securities are his and his alone. Other investors would obviously have different estimates of these variables. However, the estimates might not vary much because all investors would be forming expectations from the same data about past price movements and other publicly available information.

Financial economists often imagine a world where all investors possess the *same* estimates of expected returns, variances, and covariances. Though this can never be literally true, it can be thought of as a useful simplifying assumption in a world where investors have access to similar sources of information. This assumption is called **homogeneous expectations**.[13]

If all investors had homogeneous expectations, Figure 10.9 would be the same for all individuals. That is, all investors would sketch out the same efficient set of risky assets because they would be working with the same inputs. This efficient set of risky assets is

[13] The assumption of homogeneous expectations states that all investors have the same beliefs concerning returns, variances, and covariances. It does not say that all investors have the same aversion to risk.

represented by the curve *XAY*. Because the same risk-free rate would apply to everyone, all investors would view point *A* as the portfolio of risky assets to be held.

This point *A* takes on great importance because all investors would purchase the risky securities that it represents. Investors with a high degree of risk aversion might combine *A* with an investment in the riskless asset, achieving point *4*, for example. Others with low aversion to risk might borrow to achieve, say, point *5*. Because this is a very important conclusion, we restate it:

In a world with homogeneous expectations, all investors would hold the portfolio of risky assets represented by point *A*.

If all investors choose the same portfolio of risky assets, it is possible to determine what that portfolio is. Common sense tells us that it is a market value weighted portfolio of all existing securities. It is the **market portfolio**.

In practice, economists use a broad-based index such as the Standard & Poor's (S&P) 500 as a proxy for the market portfolio. Of course all investors do not hold the same portfolio in practice. However, we know that many investors hold diversified portfolios, particularly when mutual funds or pension funds are included. A broad-based index is a good proxy for the highly diversified portfolios of many investors.

Definition of Risk When Investors Hold the Market Portfolio

The previous section states that many investors hold diversified portfolios similar to broad-based indexes. This result allows us to be more precise about the risk of a security in the context of a diversified portfolio.

Researchers have shown that the best measure of the risk of a security in a large portfolio is the *beta* of the security. We illustrate beta by an example.

EXAMPLE 10.4

Beta Consider the following possible returns both on the stock of Jelco, Inc., and on the market:

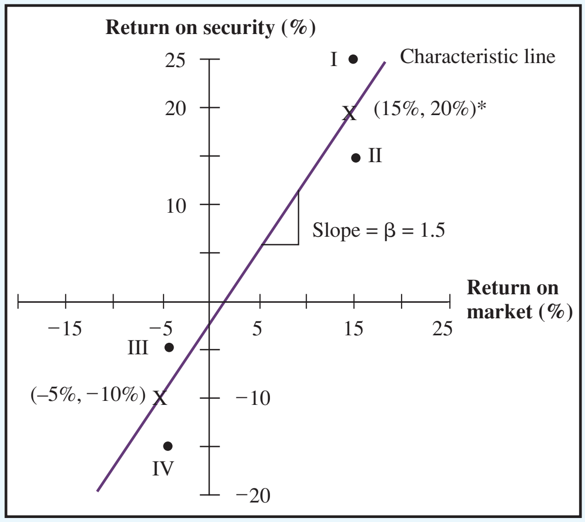
State	Type of Economy	Return on Market (percent)	Return on Jelco, Inc. (percent)
I	Bull	15	25
II	Bull	15	15
III	Bear	−5	−5
IV	Bear	−5	−15

Though the return on the market has only two possible outcomes (15% and −5%), the return on Jelco has four possible outcomes. It is helpful to consider the expected return on a security for a given return on the market. Assuming each state is equally likely, we have:

Type of Economy	Return on Market (percent)	Expected Return on Jelco, Inc. (percent)
Bull	15%	$20\% = 25\% \times \frac{1}{2} + 15\% \times \frac{1}{2}$
Bear	−5%	$-10\% = -5\% \times \frac{1}{2} + (-15\%) \times \frac{1}{2}$

(continued)

Figure 10.10 **Performance of Jelco, Inc., and the Market Portfolio**



The two points marked X represent the expected return on Jelco for each possible outcome of the market portfolio. The expected return on Jelco is positively related to the return on the market. Because the slop is 1.5, we say that Jelco's beta is 1.5. Beta measures the responsiveness of the security's return to movement in the market.
*** (15%, 20%) refers to the point where the return on the market is 15 percent and the return on the security is 20 percent.**

Jelco, Inc., responds to market movements because its expected return is greater in bullish states than in bearish states. We now calculate exactly how responsive the security is to market movements. The market's return in a bullish economy is 20 percent [= 15% − (−5%)] greater than the market's return in a bearish economy. However, the expected return on Jelco in a bullish economy is 30 percent [= 20% − (−10%)] greater than its expected return in a bearish state. Thus Jelco, Inc., has a responsiveness coefficient of 1.5 (= 30%/20%).

This relationship appears in Figure 10.10. The returns for both Jelco and the market in each state are plotted as four points. In addition, we plot the expected return on the security for each of the two possible returns on the market. These two points, each of which we designate by an X, are joined by a line called the **characteristic line** of the security. The slope of the line is 1.5, the number calculated in the previous paragraph. This responsiveness coefficient of 1.5 is the **beta** of Jelco.

The interpretation of beta from Figure 10.10 is intuitive. The graph tells us that the returns of Jelco are magnified 1.5 times over those of the market. When the market does well, Jelco's stock is expected to do even better. When the market does poorly, Jelco's stock is expected to do even worse. Now imagine an individual with a portfolio near that of the market who is considering the addition of Jelco to her portfolio. Because of Jelco's *magnification factor* of 1.5, she will view this stock as contributing much to the risk of the portfolio. (We will show shortly that the beta of the average security in the market is 1.) Jelco contributes more to the risk of a large, diversified portfolio than does an average security because Jelco is more responsive to movements in the market.

Further insight can be gleaned by examining securities with negative betas. One should view these securities as either hedges or insurance policies. The security is expected to do well when the market does poorly and vice versa. Because of this, adding a negative-beta security to a large, diversified portfolio actually reduces the risk of the portfolio.[14]

[14]Unfortunately, empirical evidence shows that virtually no stocks have negative betas.

Table 10.7

Estimates of Beta for Selected Individual Stocks

Stock	Beta
McGraw-Hill Co.	.52
3M	.66
General Electric	.83
Bed, Bath & Beyond	.98
Dell	1.22
Home Depot	1.44
eBay	2.06
Computer Associates	2.58

The beta is defined as $\text{Cov}(R_i, R_M)/\text{Var}(R_M)$, where $\text{Cov}(R_i, R_M)$ is the covariance of the return on an individual stock, R_i, and the return on the market, R_M. $\text{Var}(R_M)$ is the variance of the return on the market, R_M.

Table 10.7 presents empirical estimates of betas for individual securities. As can be seen, some securities are more responsive to the market than others. For example, eBay has a beta of 2.06. This means that for every 1 percent movement in the market,[15] eBay is expected to move 2.06 percent in the same direction. Conversely, General Electric has a beta of only 0.83. This means that for every 1 percent movement in the market, General Electric is expected to move 0.83 percent in the same direction.

We can summarize our discussion of beta by saying this:

Beta measures the responsiveness of a security to movements in the market portfolio.

The Formula for Beta

Our discussion so far has stressed the intuition behind beta. The actual definition of beta is:

$$\beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma^2(R_M)} \qquad \textbf{(10.15)}$$

where $\text{Cov}(R_i, R_M)$ is the covariance between the return on asset i and the return on the market portfolio and $\sigma^2(R_M)$ is the variance of the market.

One useful property is that the average beta across all securities, when weighted by the proportion of each security's market value to that of the market portfolio, is 1. That is:

$$\sum_{i=1}^{N} X_i \beta_i = 1 \qquad \textbf{(10.16)}$$

where X_i is the proportion of security i's market value to that of the entire market and N is the number of securities in the market.

Equation 10.16 is intuitive, once you think about it. If you weight all securities by their market values, the resulting portfolio is the market. By definition, the beta of the market portfolio is 1. That is, for every 1 percent movement in the market, the market must move 1 percent—*by definition*.

A Test

We have put these questions on past corporate finance examinations:

1. What sort of investor rationally views the variance (or standard deviation) of an individual security's return as the security's proper measure of risk?
2. What sort of investor rationally views the beta of a security as the security's proper measure of risk?

[15] In Table 10.7, we use the Standard & Poor's 500 Index as a proxy for the market portfolio.

A good answer might be something like the following:

> A rational, risk-averse investor views the variance (or standard deviation) of her portfolio's return as the proper measure of the risk of her portfolio. If for some reason the investor can hold only one security, the variance of that security's return becomes the variance of the portfolio's return. Hence, the variance of the security's return is the security's proper measure of risk.
>
> If an individual holds a diversified portfolio, she still views the variance (or standard deviation) of her portfolio's return as the proper measure of the risk of her portfolio. However, she is no longer interested in the variance of each individual security's return. Rather, she is interested in the contribution of an individual security to the variance of the portfolio.

Under the assumption of homogeneous expectations, all individuals hold the market portfolio. Thus, we measure risk as the contribution of an individual security to the variance of the market portfolio. This contribution, when standardized properly, is the beta of the security. Although few investors hold the market portfolio exactly, many hold reasonably diversified portfolios. These portfolios are close enough to the market portfolio so that the beta of a security is likely to be a reasonable measure of its risk.

10.9 Relationship between Risk and Expected Return (CAPM)

It is commonplace to argue that the expected return on an asset should be positively related to its risk. That is, individuals will hold a risky asset only if its expected return compensates for its risk. In this section, we first estimate the expected return on the stock market as a whole. Next, we estimate expected returns on individual securities.

Expected Return on Market

Economists frequently argue that the expected return on the market can be represented as:

$$\overline{R}_M = R_F + \text{Risk premium}$$

In words, the expected return on the market is the sum of the risk-free rate plus some compensation for the risk inherent in the market portfolio. Note that the equation refers to the *expected* return on the market, not the actual return in a particular month or year. Because stocks have risk, the actual return on the market over a particular period can, of course, be below R_F or can even be negative.

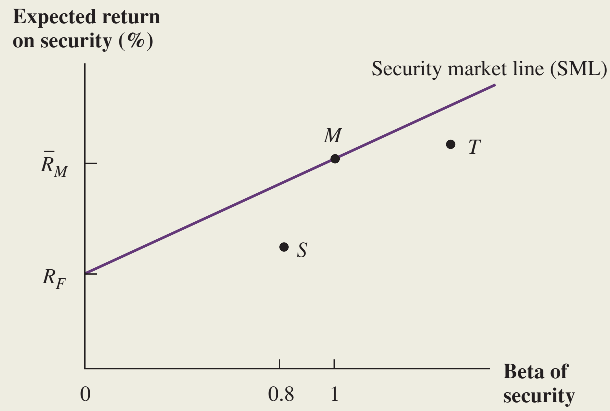
Because investors want compensation for risk, the risk premium is presumably positive. But exactly how positive is it? It is generally argued that the place to start looking for the risk premium in the future is the average risk premium in the past. As reported in Chapter 9, Ibbotson and Sinquefield found that the average return on large-company common stocks was 12.3 percent over 1926–2005. The average risk-free rate over the same interval was 3.8 percent. Thus, the average difference between the two was 8.5 percent (=12.3% − 3.8%). Financial economists find this to be a useful estimate of the difference to occur in the future.

For example, if the risk-free rate, estimated by the current yield on a one-year Treasury bill, is 1 percent, the expected return on the market is:

$$9.5\% = 1\% + 8.5\%$$

Figure 10.11

Relationship between Expected Return on an individual Security and Beta of the Security



The Security Market Line (SML) is the graphical depiction of the capital asset pricing model (CAPM).
The expected return on a stock with a beta of 0 is equal to the risk-free rate.
The expected return on a stock with a beta of 1 is equal to the expected return on the market.

Of course, the future equity risk premium could be higher or lower than the historical equity risk premium. This could be true if future risk is higher or lower than past risk or if individual risk aversions are higher or lower than those of the past.

Expected Return on Individual Security

Now that we have estimated the expected return on the market as a whole, what is the expected return on an individual security? We have argued that the beta of a security is the appropriate measure of risk in a large, diversified portfolio. Because most investors are diversified, the expected return on a security should be positively related to its beta. This is illustrated in Figure 10.11.

Actually, economists can be more precise about the relationship between expected return and beta. They posit that under plausible conditions the relationship between expected return and beta can be represented by the following equation:[16]

Capital Asset Pricing Model

$$\bar{R} = R_F + \beta \times (\bar{R}_M - R_F) \qquad \textbf{(10.17)}$$

Expected return on a security	=	Risk-free rate	+	Beta of the security	×	Difference between expected return on market and risk-free rate

This formula, which is called the **capital asset pricing model** (or CAPM for short), implies that the expected return on a security is linearly related to its beta. Because the average return on the market has been higher than the average risk-free rate over long periods of time, $\bar{R}_M - R_F$ is presumably positive. Thus, the formula implies that the expected return on a security is *positively* related to its beta. The formula can be illustrated by assuming a few special cases:

- *Assume that* $\beta = 0$. Here $\bar{R} = R_F$—that is, the expected return on the security is equal to the risk-free rate. Because a security with zero beta has no relevant risk, its expected return should equal the risk-free rate.

[16]This relationship was first proposed independently by John Lintner and William F. Sharpe.

- *Assume that* $\beta = 1$. Equation 10.17 reduces to $\overline{R} = \overline{R}_M$. That is, the expected return on the security is equal to the expected return on the market. This makes sense because the beta of the market portfolio is also 1.

Equation 10.17 can be represented graphically by the upward-sloping line in Figure 10.11. Note that the line begins at R_F and rises to $\overline{R}_M$ when beta is 1. This line is frequently called the **security market line** (SML).

As with any line, the SML has both a slope and an intercept. R_F, the risk-free rate, is the intercept. Because the beta of a security is the horizontal axis, $R_M - R_F$ is the slope. The line will be upward-sloping as long as the expected return on the market is greater than the risk-free rate. Because the market portfolio is a risky asset, theory suggests that its expected return is above the risk-free rate. As mentioned, the empirical evidence of the previous chapter showed that the average return per year on the market portfolio (for large-company stocks as an example) over the past 80 years was 8.5 percent above the risk-free rate.

EXAMPLE 10.5

The stock of Aardvark Enterprises has a beta of 1.5 and that of Zebra Enterprises has a beta of 0.7. The risk-free rate is assumed to be 3 percent, and the difference between the expected return on the market and the risk-free rate is assumed to be 8.0 percent. The expected returns on the two securities are

Expected Return for Aardvark

$$15.0\% = 3\% + 1.5 \times 8.0\% \tag{10.18}$$

Expected Return for Zebra

$$8.6\% = 3\% + 0.7 \times 8.0\%$$

Three additional points concerning the CAPM should be mentioned:

1. *Linearity*: The intuition behind an upwardly sloping curve is clear. Because beta is the appropriate measure of risk, high-beta securities should have an expected return above that of low-beta securities. However, both Figure 10.11 and Equation 10.17 show something more than an upwardly sloping curve: The relationship between expected return and beta corresponds to a *straight* line.

   It is easy to show that the line of Figure 10.11 is straight. To see this, consider security *S* with, say, a beta of 0.8. This security is represented by a point below the security market line in the figure. Any investor could duplicate the beta of security *S* by buying a portfolio with 20 percent in the risk-free asset and 80 percent in a security with a beta of 1. However, the homemade portfolio would itself lie on the SML. In other words, the portfolio dominates security *S* because the portfolio has a higher expected return and the same beta.

   Now consider security *T* with, say, a beta greater than 1. This security is also below the SML in Figure 10.11. Any investor could duplicate the beta of security *T* by borrowing to invest in a security with a beta of 1. This portfolio must also lie on the SML, thereby dominating security *T*.

   Because no one would hold either *S* or *T*, their stock prices would drop. This price adjustment would raise the expected returns on the two securities. The price adjustment would continue until the two securities lay on the security market line. The preceding example considered two overpriced stocks and a straight SML. Securities lying above the SML are *underpriced*. Their prices must rise until their expected returns lie on the line. If the SML is itself curved, many stocks would be mispriced. In equilibrium, all securities would be held only when prices changed so that the SML became straight. In other words, linearity would be achieved.

2. *Portfolios as well as securities*: Our discussion of the CAPM considered individual securities. Does the relationship in Figure 10.11 and Equation 10.17 hold for portfolios as well?

   Yes. To see this, consider a portfolio formed by investing equally in our two securities from Example 10.5, Aardvark and Zebra. The expected return on the portfolio is:

   Expected Return on Portfolio

   $$11.8\% = 0.5 \times 15.0\% + 0.5 \times 8.6\% \qquad \textbf{(10.19)}$$

   The beta of the portfolio is simply a weighted average of the betas of the two securities. Thus, we have:

   Beta of Portfolio

   $$1.1 = 0.5 \times 1.5 + 0.5 \times 0.7$$

   Under the CAPM, the expected return on the portfolio is

   $$11.8\% = 3\% + 1.1 \times 8.0\% \qquad \textbf{(10.20)}$$

   Because the expected return in Equation 10.19 is the same as the expected return in Equation 10.20, the example shows that the CAPM holds for portfolios as well as for individual securities.

3. *A potential confusion*: Students often confuse the SML in Figure 10.11 with line *II* in Figure 10.9. Actually, the lines are quite different. Line *II* traces the efficient set of portfolios formed from both risky assets and the riskless asset. Each point on the line represents an entire portfolio. Point *A* is a portfolio composed entirely of risky assets. Every other point on the line represents a portfolio of the securities in *A* combined with the riskless asset. The axes on Figure 10.9 are the expected return on a *portfolio* and the standard deviation of a *portfolio*. Individual securities do not lie along line *II*.

   The SML in Figure 10.11 relates expected return to beta. Figure 10.11 differs from Figure 10.9 in at least two ways. First, beta appears in the horizontal axis of Figure 10.11, but standard deviation appears in the horizontal axis of Figure 10.9. Second, the SML in Figure 10.11 holds both for all individual securities and for all possible portfolios, whereas line *II* in Figure 10.9 holds only for efficient portfolios.

We stated earlier that, under homogeneous expectations, point *A* in Figure 10.9 becomes the market portfolio. In this situation, line *II* is referred to as the **capital market line** (CML).

Summary and Conclusions

This chapter set forth the fundamentals of modern portfolio theory. Our basic points are these:

1. This chapter showed us how to calculate the expected return and variance for individual securities, and the covariance and correlation for pairs of securities. Given these statistics, the expected return and variance for a portfolio of two securities *A* and *B* can be written as:

$$\text{Expected return on portfolio} = X_A\bar{R}_A + X_B\bar{R}_B$$
$$\text{Var(portfolio)} = X_A^2\sigma_A^2 + 2X_AX_B\sigma_{AB} + X_B^2\sigma_B^2$$

2. In our notation, X stands for the proportion of a security in a portfolio. By varying X we can trace out the efficient set of portfolios. We graphed the efficient set for the two-asset case as a curve, pointing out that the degree of curvature or bend in the graph reflects the diversification effect: The lower the correlation between the two securities, the greater the bend. The same general shape of the efficient set holds in a world of many assets.
3. Just as the formula for variance in the two-asset case is computed from a 2×2 matrix, the variance formula is computed from an $N \times N$ matrix in the N-asset case. We showed that with a large number of assets, there are many more covariance terms than variance terms in the matrix. In fact the variance terms are effectively diversified away in a large portfolio, but the covariance terms are not. Thus, a diversified portfolio can eliminate some, but not all, of the risk of the individual securities.
4. The efficient set of risky assets can be combined with riskless borrowing and lending. In this case a rational investor will always choose to hold the portfolio of risky securities represented by point A in Figure 10.9. Then he can either borrow or lend at the riskless rate to achieve any desired point on line II in the figure.
5. The contribution of a security to the risk of a large, well-diversified portfolio is proportional to the covariance of the security's return with the market's return. This contribution, when standardized, is called the beta. The beta of a security can also be interpreted as the responsiveness of a security's return to that of the market.
6. The CAPM states that:

$$\overline{R} = R_F + \beta(\overline{R}_M - R_F)$$

In other words, the expected return on a security is positively (and linearly) related to the security's beta.

Concept Questions

1. **Diversifiable and Nondiversifiable Risks** In broad terms, why is some risk diversifiable? Why are some risks nondiversifiable? Does it follow that an investor can control the level of unsystematic risk in a portfolio, but not the level of systematic risk?
2. **Systematic versus Unsystematic Risk** Classify the following events as mostly systematic or mostly unsystematic. Is the distinction clear in every case?
   a. Short-term interest rates increase unexpectedly.
   b. The interest rate a company pays on its short-term debt borrowing is increased by its bank.
   c. Oil prices unexpectedly decline.
   d. An oil tanker ruptures, creating a large oil spill.
   e. A manufacturer loses a multimillion-dollar product liability suit.
   f. A Supreme Court decision substantially broadens producer liability for injuries suffered by product users.
3. **Expected Portfolio Returns** If a portfolio has a positive investment in every asset, can the expected return on the portfolio be greater than that on every asset in the portfolio? Can it be less than that on every asset in the portfolio? If you answer yes to one or both of these questions, give an example to support your answer.
4. **Diversification** True or false: The most important characteristic in determining the expected return of a well-diversified portfolio is the variances of the individual assets in the portfolio. Explain.
5. **Portfolio Risk** If a portfolio has a positive investment in every asset, can the standard deviation on the portfolio be less than that on every asset in the portfolio? What about the portfolio beta?
6. **Beta and CAPM** Is it possible that a risky asset could have a beta of zero? Explain. Based on the CAPM, what is the expected return on such an asset? Is it possible that a risky asset could have a negative beta? What does the CAPM predict about the expected return on such an asset? Can you give an explanation for your answer?

7. **Covariance** Briefly explain why the covariance of a security with the rest of a well-diversified portfolio is a more appropriate measure of the risk of the security than the security's variance.
8. **Beta** Consider the following quotation from a leading investment manager: "The shares of Southern Co. have traded close to $12 for most of the past three years. Since Southern's stock has demonstrated very little price movement, the stock has a low beta. Texas Instruments, on the other hand, has traded as high as $150 and as low as its current $75. Since TI's stock has demonstrated a large amount of price movement, the stock has a very high beta." Do you agree with this analysis? Explain.
9. **Risk** A broker has advised you not to invest in oil industry stocks because they have high standard deviations. Is the broker's advice sound for a risk-averse investor like yourself? Why or why not?
10. **Security Selection** Is the following statement true or false? A risky security cannot have an expected return that is less than the risk-free rate because no risk-averse investor would be willing to hold this asset in equilibrium. Explain.

Questions and Problems

BASIC
(Questions 1–20)

1. **Determining Portfolio Weights** What are the portfolio weights for a portfolio that has 70 shares of Stock *A* that sell for $40 per share and 110 shares of Stock *B* that sell for $22 per share?
2. **Portfolio Expected Return** You own a portfolio that has $1,200 invested in Stock *A* and $1,900 invested in Stock *B*. If the expected returns on these stocks are 11 percent and 16 percent, respectively, what is the expected return on the portfolio?
3. **Portfolio Expected Return** You own a portfolio that is 50 percent invested in Stock *X*, 30 percent in Stock *Y*, and 20 percent in Stock *Z*. The expected returns on these three stocks are 11 percent, 17 percent, and 14 percent, respectively. What is the expected return on the portfolio?
4. **Portfolio Expected Return** You have $10,000 to invest in a stock portfolio. Your choices are Stock *X* with an expected return of 14 percent and Stock *Y* with an expected return of 9 percent. If your goal is to create a portfolio with an expected return of 12.2 percent, how much money will you invest in Stock *X*? In Stock *Y*?
5. **Calculating Expected Return** Based on the following information, calculate the expected return:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs
Recession	.20	−.05
Normal	.50	.12
Boom	.30	.25

6. **Calculating Returns and Standard Deviations** Based on the following information, calculate the expected return and standard deviation for the two stocks:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs	
		Stock *A*	Stock *B*
Recession	.10	.06	−.20
Normal	.60	.07	.13
Boom	.30	.11	.33

7. **Calculating Returns and Standard Deviations** Based on the following information, calculate the expected return and standard deviation:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs
Depression	.10	−.045
Recession	.20	.044
Normal	.50	.120
Boom	.20	.207

8. **Calculating Expected Returns** A portfolio is invested 20 percent in Stock *G*, 70 percent in Stock *J*, and 10 percent in Stock *K*. The expected returns on these stocks are 8 percent, 15 percent, and 24 percent, respectively. What is the portfolio's expected return? How do you interpret your answer?

9. **Returns and Standard Deviations** Consider the following information:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs		
		Stock *A*	Stock *B*	Stock *C*
Boom	.70	.07	.15	.33
Bust	.30	.13	.03	−.06

a. What is the expected return on an equally weighted portfolio of these three stocks?
b. What is the variance of a portfolio invested 20 percent each in *A* and *B*, and 60 percent in *C*?

10. **Returns and Standard Deviations** Consider the following information:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs		
		Stock *A*	Stock *B*	Stock *C*
Boom	.30	.30	.45	.33
Good	.40	.12	.10	.15
Poor	.25	.01	−.15	−.05
Bust	.05	−.06	−.30	−.09

a. Your portfolio is invested 30 percent each in *A* and *C*, and 40 percent in *B*. What is the expected return of the portfolio?
b. What is the variance of this portfolio? The standard deviation?

11. **Calculating Portfolio Betas** You own a stock portfolio invested 25 percent in Stock *Q*, 20 percent in Stock *R*, 15 percent in Stock *S*, and 40 percent in Stock *T*. The betas for these four stocks are .6, 1.70, 1.15, and 1.34, respectively. What is the portfolio beta?

12. **Calculating Portfolio Betas** You own a portfolio equally invested in a risk-free asset and two stocks. If one of the stocks has a beta of 1.9 and the total portfolio is equally as risky as the market, what must the beta be for the other stock in your portfolio?

13. **Using CAPM** A stock has a beta of 1.3, the expected return on the market is 14 percent, and the risk-free rate is 5 percent. What must the expected return on this stock be?

14. **Using CAPM** A stock has an expected return of 14 percent, the risk-free rate is 4 percent, and the market risk premium is 6 percent. What must the beta of this stock be?

15. **Using CAPM** A stock has an expected return of 11 percent, its beta is .85, and the risk-free rate is 5.5 percent. What must the expected return on the market be?

16. **Using CAPM** A stock has an expected return of 17 percent, a beta of 1.9, and the expected return on the market is 11 percent. What must the risk-free rate be?

17. **Using CAPM** A stock has a beta of 1.2 and an expected return of 16 percent. A risk-free asset currently earns 5 percent.

a. What is the expected return on a portfolio that is equally invested in the two assets?
b. If a portfolio of the two assets has a beta of .75, what are the portfolio weights?
c. If a portfolio of the two assets has an expected return of 8 percent, what is its beta?
d. If a portfolio of the two assets has a beta of 2.40, what are the portfolio weights? How do you interpret the weights for the two assets in this case? Explain.

18. **Using the SML** Asset *W* has an expected return of 16 percent and a beta of 1.3. If the risk-free rate is 5 percent, complete the following table for portfolios of Asset *W* and a risk-free asset. Illustrate the relationship between portfolio expected return and portfolio beta by plotting the expected returns against the betas. What is the slope of the line that results?

Percentage of Portfolio in Asset *W*	Portfolio Expected Return	Portfolio Beta
0%		
25		
50		
75		
100		
125		
150		

19. **Reward-to-Risk Ratios** Stock *Y* has a beta of 1.50 and an expected return of 17 percent. Stock *Z* has a beta of .80 and an expected return of 10.5 percent. If the risk-free rate is 5.5 percent and the market risk premium is 7.5 percent, are these stocks correctly priced?

20. **Reward-to-Risk Ratios** In the previous problem, what would the risk-free rate have to be for the two stocks to be correctly priced?

INTERMEDIATE (Questions 21–33)

21. **Portfolio Returns** Using information from the previous chapter about capital market history, determine the return on a portfolio that is equally invested in large-company stocks and long-term government bonds. What is the return on a portfolio that is equally invested in small-company stocks and Treasury bills?

22. **CAPM** Using the CAPM, show that the ratio of the risk premiums on two assets is equal to the ratio of their betas.

23. **Portfolio Returns and Deviations** Consider the following information about three stocks:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs		
		Stock *A*	Stock *B*	Stock *C*
Boom	.4	.20	.35	.60
Normal	.4	.15	.12	.05
Bust	.2	.01	−.25	−.50

a. If your portfolio is invested 40 percent each in *A* and *B* and 20 percent in *C*, what is the portfolio expected return? The variance? The standard deviation?

b. If the expected T-bill rate is 3.80 percent, what is the expected risk premium on the portfolio?

c. If the expected inflation rate is 3.50 percent, what are the approximate and exact expected real returns on the portfolio? What are the approximate and exact expected real risk premiums on the portfolio?

24. **Analyzing a Portfolio** You want to create a portfolio equally as risky as the market, and you have $1,000,000 to invest. Given this information, fill in the rest of the following table:

Asset	Investment	Beta
Stock *A*	$200,000	.80
Stock *B*	$250,000	1.30
Stock *C*		1.50
Risk-free asset		

25. **Analyzing a Portfolio** You have $100,000 to invest in a portfolio containing Stock *X*, Stock *Y*, and a risk-free asset. You must invest all of your money. Your goal is to create a portfolio that has an expected return of 13.5 percent and that has only 70 percent of the risk of the overall market. If *X* has an expected return of 31 percent and a beta of 1.8, *Y* has an expected return of 20 percent and a beta of 1.3, and the risk-free rate is 7 percent, how much money will you invest in Stock *X*? How do you interpret your answer?

26. **Covariance and Correlation** Based on the following information, calculate the expected return and standard deviation of each of the following stocks. Assume each state of the economy is equally likely to happen. What are the covariance and correlation between the returns of the two stocks?

State of Economy	Return on Stock *A*	Return on Stock *B*
Bear	.063	−.037
Normal	.105	.064
Bull	.156	.253

27. **Covariance and Correlation** Based on the following information, calculate the expected return and standard deviation for each of the following stocks. What are the covariance and correlation between the returns of the two stocks?

State of Economy	Probability of State of Economy	Return on Stock *J*	Return on Stock *K*
Bear	.25	−.020	.050
Normal	.60	.092	.062
Bull	.15	.154	.074

28. **Portfolio Standard Deviation** Security *F* has an expected return of 12 percent and a standard deviation of 34 percent per year. Security *G* has an expected return of 18 percent and a standard deviation of 50 percent per year.

a. What is the expected return on a portfolio composed of 30 percent of security *F* and 70 percent of security *G*?

b. If the correlation between the returns of security F and security G is .2, what is the standard deviation of the portfolio described in part (a)?

29. Portfolio Standard Deviation Suppose the expected returns and standard deviations of stocks A and B are $E(R_A) = .15$, $E(R_B) = .25$, $\sigma_A = .40$, and $\sigma_B = .65$, respectively.

a. Calculate the expected return and standard deviation of a portfolio that is composed of 40 percent A and 60 percent B when the correlation between the returns on A and B is .5.

b. Calculate the standard deviation of a portfolio that is composed of 40 percent A and 60 percent B when the correlation coefficient between the returns on A and B is $-.5$.

c. How does the correlation between the returns on A and B affect the standard deviation of the portfolio?

30. Correlation and Beta You have been provided the following data about the securities of three firms, the market portfolio, and the risk-free asset:

Security	Expected Return	Standard Deviation	Correlation*	Beta
Firm *A*	.13	.38	(i)	.9
Firm *B*	.16	(ii)	.4	1.1
Firm *C*	.25	.65	.35	(iii)
The market portfolio	.15	.20	(iv)	(v)
The risk-free asset	.05	(vi)	(vii)	(viii)

*With the market portfolio.

a. Fill in the missing values in the table.

b. Is the stock of Firm A correctly priced according to the capital asset pricing model (CAPM)? What about the stock of Firm B? Firm C? If these securities are not correctly priced, what is your investment recommendation for someone with a well-diversified portfolio?

31. CML The market portfolio has an expected return of 12 percent and a standard deviation of 10 percent. The risk-free rate is 5 percent.

a. What is the expected return on a well-diversified portfolio with a standard deviation of 7 percent?

b. What is the standard deviation of a well-diversified portfolio with an expected return of 20 percent?

32. Beta and CAPM A portfolio that combines the risk-free asset and the market portfolio has an expected return of 12 percent and a standard deviation of 18 percent. The risk-free rate is 5 percent, and the expected return on the market portfolio is 14 percent. Assume the capital asset pricing model holds. What expected rate of return would a security earn if it had a .45 correlation with the market portfolio and a standard deviation of 40 percent?

33. Beta and CAPM Suppose the risk-free rate is 6.3 percent and the market portfolio has an expected return of 14.8 percent. The market portfolio has a variance of .0498. Portfolio Z has a correlation coefficient with the market of .45 and a variance of .1783. According to the capital asset pricing model, what is the expected return on portfolio Z?

CHALLENGE (Questions 34–39)

34. Systematic versus Unsystematic Risk Consider the following information about Stocks I and II:

State of Economy	Probability of State of Economy	Rate of Return If State Occurs	
		Stock *I*	Stock *II*
Recession	.15	.09	−.30
Normal	.70	.42	.12
Irrational exuberance	.15	.26	.44

The market risk premium is 10 percent, and the risk-free rate is 4 percent. Which stock has the most systematic risk? Which one has the most unsystematic risk? Which stock is "riskier"? Explain.

35. SML Suppose you observe the following situation:

Security	Beta	Expected Return
Pete Corp.	1.3	.23
Repete Co.	.6	.13

Assume these securities are correctly priced. Based on the CAPM, what is the expected return on the market? What is the risk-free rate?

36. Covariance and Portfolio Standard Deviation There are three securities in the market. The following chart shows their possible payoffs:

State	Probability of Outcome	Return on Security 1	Return on Security 2	Return on Security 3
1	.10	.25	.25	.10
2	.40	.20	.15	.15
3	.40	.15	.20	.20
4	.10	.10	.10	.25

a. What are the expected return and standard deviation of each security?
b. What are the covariances and correlations between the pairs of securities?
c. What are the expected return and standard deviation of a portfolio with half of its funds invested in security 1 and half in security 2?
d. What are the expected return and standard deviation of a portfolio with half of its funds invested in security 1 and half in security 3?
e. What are the expected return and standard deviation of a portfolio with half of its funds invested in security 2 and half in security 3?
f. What do your answers in parts (a), (c), (d), and (e) imply about diversification?

37. SML Suppose you observe the following situation:

State of Economy	Probability of State	Return If State Occurs	
		Stock *A*	Stock *B*
Bust	.25	−.10	−.30
Normal	.50	.10	.05
Boom	.25	.20	.40

a. Calculate the expected return on each stock.
b. Assuming the capital asset pricing model holds and stock *A*'s beta is greater than stock *B*'s beta by .25, what is the expected market risk premium?

38. **Standard Deviation and Beta** There are two stocks in the market, stock *A* and stock *B*. The price of stock *A* today is $50. The price of stock *A* next year will be $40 if the economy is in a recession, $55 if the economy is normal, and $60 if the economy is expanding. The probabilities of recession, normal times, and expansion are .1, .8, and .1, respectively. Stock *A* pays no dividends and has a correlation of .8 with the market portfolio. Stock *B* has an expected return of 9 percent, a standard deviation of 12 percent, a correlation with the market portfolio of .2, and a correlation with stock *A* of .6. The market portfolio has a standard deviation of 10 percent. Assume the CAPM holds.
   - **a.** If you are a typical, risk-averse investor with a well-diversified portfolio, which stock would you prefer? Why?
   - **b.** What are the expected return and standard deviation of a portfolio consisting of 70 percent of stock *A* and 30 percent of stock *B*?
   - **c.** What is the beta of the portfolio in part (b)?

39. **Minimum Variance Portfolio** Assume stocks *A* and *B* have the following characteristics:

Stock	Expected Return (%)	Standard Deviation (%)
A	5	10
B	10	20

The covariance between the returns on the two stocks is .001.
   - **a.** Suppose an investor holds a portfolio consisting of only stock *A* and stock *B*. Find the portfolio weights, X_A and X_B, such that the variance of her portfolio is minimized. (*Hint*: Remember that the sum of the two weights must equal 1.)
   - **b.** What is the expected return on the minimum variance portfolio?
   - **c.** If the covariance between the returns on the two stocks is −.02, what are the minimum variance weights?
   - **d.** What is the variance of the portfolio in part (c)?

S&P Problem

STANDARD &POOR'S

www.mhhe.com/edumarketinsight

1. **Using CAPM** You can find estimates of beta for companies under the "Mthly. Val. Data" link. Locate the beta for Amazon.com (AMZN) and Dow Chemical (DOW). How has the beta for each of these companies changed over the period reported? Using the historical risk-free rate and market risk premium found in the chapter, calculate the expected return for each company based on the most recent beta. Is the expected return for each company what you would expect? Why or why not?

Appendix 10A Is Beta Dead?

To access Appendix 10A, please go to www.mhhe.com/rwj.

A Job at East Coast Yachts, Part 2

You are discussing your 401(k) with Dan Ervin when he mentions that Sarah Brown, a representative from Bledsoe Financial Services, is visiting East Coast Yachts today. You decide that you should meet with Sarah, so Dan sets up an appointment for you later in the day.

When you sit down with Sarah, she discusses the various investment options available in the company's 401(k) account. You mention to Sarah that you researched East Coast Yachts before you accepted your new job. You are confident in management's ability to lead the company. Analysis of the company has led to your belief that the company is growing and will achieve a greater market share in the future. You also feel you should support your employer. Given these considerations, along with the fact that you are a conservative investor, you are leaning toward investing 100 percent of your 401(k) account in East Coast Yachts.

Assume the risk-free rate is the historical average risk-free rate (in Chapter 9). The correlation between the Bledsoe bond fund and large-cap stock fund is .27. Note that the spreadsheet graphing and "solver" functions may assist you in answering the following questions.

1. Considering the effects of diversification, how should Sarah respond to the suggestion that you invest 100 percent of your 401(k) account in East Coast Yachts stock?
2. Sarah's response to investing your 401(k) account entirely in East Coast Yachts stock has convinced you that this may not be the best alternative. Because you are a conservative investor, you tell Sarah that a 100 percent investment in the bond fund may be the best alternative. Is it?
3. Using the returns for the Bledsoe Large-Cap Stock Fund and the Bledsoe Bond Fund, graph the opportunity set of feasible portfolios.
4. After examining the opportunity set, you notice that you can invest in a portfolio consisting of the bond fund and the large-cap stock fund that will have exactly the same standard deviation as the bond fund. This portfolio will also have a greater expected return. What are the portfolio weights and expected return of this portfolio?
5. Examining the opportunity set, notice there is a portfolio that has the lowest standard deviation. This is the minimum variance portfolio. What are the portfolio weights, expected return, and standard deviation of this portfolio? Why is the minimum variance portfolio important?
6. A measure of risk-adjusted performance that is often used is the Sharpe ratio. The Sharpe ratio is calculated as the risk premium of an asset divided by its standard deviation. The portfolio with the highest possible Sharpe ratio on the opportunity set is called the Sharpe optimal portfolio. What are the portfolio weights, expected return, and standard deviation of the Sharpe optimal portfolio? How does the Sharpe ratio of this portfolio compare to the Sharpe ratios of the bond fund and the large-cap stock fund? Do you see a connection between the Sharpe optimal portfolio and the CAPM? What is the connection?

Appendix 10A Is Beta Dead?

The capital asset pricing model represents one of the most important advances in financial economics. It is clearly useful for investment purposes because it shows how the expected return on an asset is related to its beta. In addition, we will show in Chapter 12 that it is useful in corporate finance because the discount rate on a project is a function of the project's beta. However, never forget that, as with any other model, the CAPM is not revealed truth but, rather, a construct to be empirically tested.

The first empirical tests of the CAPM occurred over 20 years ago and were quite supportive. Using data from the 1930s to the 1960s, researchers showed that the average return on a portfolio of stocks was positively related to the beta of the portfolio,[1] a finding consistent with the CAPM. Though some evidence in these studies was less consistent with the CAPM,[2] financial economists were quick to embrace the CAPM following these empirical papers.

Although a large body of empirical work developed in the following decades, often with varying results, the CAPM was not seriously called into question until the 1990s. Two papers by Fama and French[3] (yes, the same Fama whose joint paper in 1973 with James MacBeth supported the CAPM) present evidence inconsistent with the model. Their work has received a great deal of attention, both in academic circles and in the popular press, with newspaper articles displaying headlines such as "Beta Is Dead!" These papers make two related points. First, they conclude that the relationship between average return and beta is weak over the period from 1941 to 1990 and virtually nonexistent from 1963 to 1990. Second, they argue that the average return on a security is negatively related to both the firm's price-earnings (P/E) ratio and the firm's market-to-book (M/B) ratio. These contentions, if confirmed by other research, would be quite damaging to the CAPM. After all, the CAPM states that the expected returns on stocks should be related *only* to beta, and not to other factors such as P/E and M/B.

However, a number of researchers have criticized the Fama–French papers. We avoid an in-depth discussion of the fine points of the debate, but we mention a few issues. First, although Fama and French cannot reject the hypothesis that average returns are unrelated to beta, we can also not reject the hypothesis that average returns are related to beta exactly as specified by the CAPM. In other words, although 50 years of data seem like a lot, they may simply not be enough to test the CAPM properly. Second, the result with P/E and M/B may be due to a statistical fallacy called a hindsight bias.[4] Third, P/E and M/B are merely two of an infinite number of possible factors. Thus, the relationship between average return and both P/E and M/B may be spurious, being nothing more than the result of data mining.

[1]Perhaps the two most well-known papers were Fischer Black, Michael C. Jensen, and Myron S. Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in M. Jensen, ed., *Studies in the Theory of Capital Markets* (New York: Praeger, 1972), and Eugene F. Fama and James MacBeth, "Risk, Return and Equilibrium: Some Empirical Tests," *Journal of Political Economy* 8 (1973), pp. 607–36.

[2]For example, the studies suggest that the average return on a zero-beta portfolio is above the risk-free rate, a finding inconsistent with the CAPM.

[3]Eugene F. Fama and Kenneth R. French, "The Cross-Section of Expected Stock Returns," *Journal of Finance* 47 (1992), pp. 427–66, and E. F. Fama and K. R. French, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* 17 (1993), pp. 3–56.

[4]For example, see William J. Breen and Robert A. Koraczyk, "On Selection Biases in Book-to-Market Based Tests of Asset Pricing Models," unpublished paper. Northwestern University, November 1993; and S. P. Kothari, Jay Shanken, and Richard G. Sloan, "Another Look at the Cross-Section of Expected Stock Returns," *Journal of Finance* (March 1995).

Fourth, average returns are positively related to beta over the period from 1927 to the present. There appears to be no compelling reason for emphasizing a shorter period than this one. Fifth, average returns are actually positively related to beta over shorter periods when annual data, rather than monthly data, are used to estimate beta.[5] There appears to be no compelling reason for preferring either monthly data over annual data or vice versa. Thus we believe that although the results of Fama and French are quite intriguing, they cannot be viewed as the final word.

[5]Points 4 and 5 are addressed in the Kothari, Shanken, and Sloan paper.