CHAPTER

4

Discounted Cash Flow Valuation

PART TWO

What do baseball players Paul Konerko, A.J. Burnett, and Ramon Hernandez have in common? All three athletes signed big contracts at the end of 2005. The contract values were reported as $60 million, $55 million, and $27.5 million, respectively. But reported numbers like these are often misleading. For example, in December 2005, Ramon Hernandez signed with the Baltimore Orioles. His contract called for salaries of $4.5 million, $6.5 million, $7.5 million, and $8 million per year over the next four years (plus a guaranteed minimum $1 million in 2010, for a total of $27.5 million). Not bad, especially for someone who makes a living using the "tools of ignorance" (jock jargon for catcher's equipment).

A closer look at the numbers shows that Paul, A.J., and Ramon did pretty well, but nothing like the quoted figures. Using A.J.'s contract as an example, although the value was reported to be $55 million, it was actually payable over several years. It consisted of a $6 million signing bonus plus $49 million in future salary and bonuses. The $49 million was to be distributed as $1 million in 2006 and $12 million per year for 2007 through 2010. Because the payments were spread out over time, we must consider the time value of money, which means his contract was worth less than reported. How much did he really get? This chapter gives you the "tools of knowledge" to answer this question.

4.1 Valuation: The One-Period Case

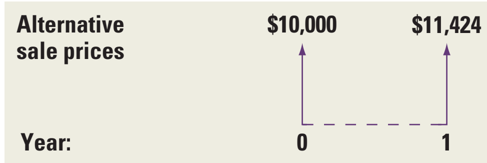
Keith Vaughn is trying to sell a piece of raw land in Alaska. Yesterday he was offered $10,000 for the property. He was about ready to accept the offer when another individual offered him $11,424. However, the second offer was to be paid a year from now. Keith has satisfied himself that both buyers are honest and financially solvent, so he has no fear that the offer he selects will fall through. These two offers are pictured as cash flows in Figure 4.1. Which offer should Keith choose?

Mike Tuttle, Keith's financial adviser, points out that if Keith takes the first offer, he could invest the $10,000 in the bank at an insured rate of 12 percent. At the end of one year, he would have:

$$\underbrace{\$10{,}000}_{\substack{\text{Return of}\\ \text{principal}}} + \underbrace{(0.12 \times \$10{,}000)}_{\text{Interest}} = \$10{,}000 \times 1.12 = \$11{,}200$$

Figure 4.1

Cash Flow for Keith Vaughn's Sale



Because this is less than the $11,424 Keith could receive from the second offer, Mike recommends that he take the latter. This analysis uses the concept of **future value (FV)** or **compound value**, which is the value of a sum after investing over one or more periods. The compound or future value of $10,000 at 12 percent is $11,200.

An alternative method employs the concept of **present value (PV)**. One can determine present value by asking the following question: How much money must Keith put in the bank today so that he will have $11,424 next year? We can write this algebraically as:

$$\text{PV} \times 1.12 = \$11{,}424$$

We want to solve for PV, the amount of money that yields $11,424 if invested at 12 percent today. Solving for PV, we have:

$$\text{PV} = \frac{\$11{,}424}{1.12} = \$10{,}200$$

The formula for PV can be written as follows:

Present Value of Investment:

$$\text{PV} = \frac{C_1}{1 + r} \qquad \textbf{(4.1)}$$

where C_1 is cash flow at date 1 and r is the rate of return that Keith Vaughn requires on his land sale. It is sometimes referred to as the *discount rate.*

Present value analysis tells us that a payment of $11,424 to be received next year has a present value of $10,200 today. In other words, at a 12 percent interest rate, Keith is indifferent between $10,200 today or $11,424 next year. If you gave him $10,200 today, he could put it in the bank and receive $11,424 next year.

Because the second offer has a present value of $10,200, whereas the first offer is for only $10,000, present value analysis also indicates that Keith should take the second offer. In other words, both future value analysis and present value analysis lead to the same decision. As it turns out, present value analysis and future value analysis must always lead to the same decision.

As simple as this example is, it contains the basic principles that we will be working with over the next few chapters. We now use another example to develop the concept of net present value.

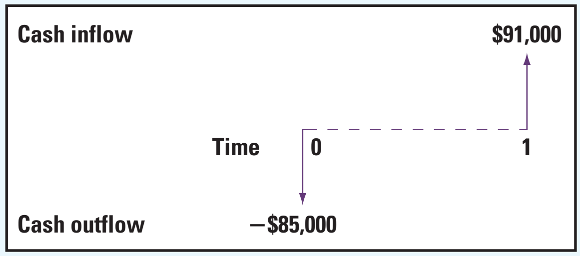
EXAMPLE 4.1

Present Value Lida Jennings, a financial analyst at Kaufman & Broad, a leading real estate firm, is thinking about recommending that Kaufman & Broad invest in a piece of land that costs $85,000. She is certain that next year the land will be worth $91,000, a sure $6,000 gain. Given that the guaranteed interest rate in the bank is 10 percent, should Kaufman & Broad undertake the investment in land? Ms. Jennings's choice is described in Figure 4.2 with the cash flow time chart.

A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $85,000 in the land, she will have $91,000 available next year. Suppose, instead,

(*continued*)

Figure 4.2 **Cash Flows for Land Investment**



that Kaufman & Broad puts the same $85,000 into the bank. At the interest rate of 10 percent, this $85,000 would grow to:

$$(1 + .10) \times \$85{,}000 = \$93{,}500$$

next year.

It would be foolish to buy the land when investing the same $85,000 in the financial market would produce an extra $2,500 (that is, $93,500 from the bank minus $91,000 from the land investment). This is a future value calculation.

Alternatively, she could calculate the present value of the sale price next year as:

$$\text{Present value} = \frac{\$91{,}000}{1.10} = \$82{,}727.27$$

Because the present value of next year's sales price is less than this year's purchase price of $85,000, present value analysis also indicates that she should not recommend purchasing the property.

Frequently, businesspeople want to determine the exact *cost* or *benefit* of a decision. In Example 4.1, the decision to buy this year and sell next year can be evaluated as:

$$\underset{}{-\$2{,}273} = \underset{\substack{\text{Cost of land}\\\text{today}}}{-\$85{,}000} + \underset{\substack{\text{Present value of}\\\text{next year's sales price}}}{\frac{\$91{,}000}{1.10}}$$

The formula for NPV can be written as follows:

Net Present Value of Investment:

$$\text{NPV} = -\text{Cost} + \text{PV} \qquad \textbf{(4.2)}$$

Equation 4.2 says that the value of the investment is −$2,273, after stating all the benefits and all the costs as of date 0. We say that −$2,273 is the **net present value** (NPV) of the investment. That is, NPV is the present value of future cash flows minus the present value of the cost of the investment. Because the net present value is negative, Lida Jennings should not recommend purchasing the land.

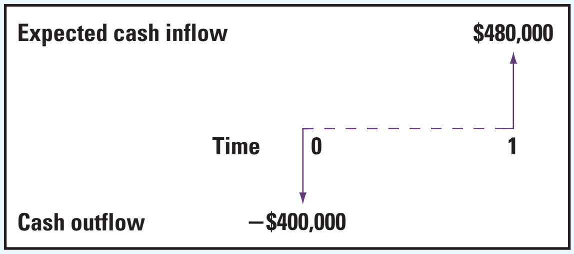
Both the Vaughn and the Jennings examples deal with perfect certainty. That is, Keith Vaughn knows with perfect certainty that he could sell his land for $11,424 next year. Similarly, Lida Jennings knows with perfect certainty that Kaufman & Broad could receive $91,000 for selling its land. Unfortunately, businesspeople frequently do not know future cash flows. This uncertainty is treated in the next example.

EXAMPLE 4.2

Uncertainty and Valuation Professional Artworks, Inc., is a firm that speculates in modern paintings. The manager is thinking of buying an original Picasso for $400,000 with the intention of selling it at the end of one year. The manager expects that the painting will be worth $480,000 in one year. The relevant cash flows are depicted in Figure 4.3.

Of course, this is only an expectation—the painting could be worth more or less than $480,000. Suppose the guaranteed interest rate granted by banks is 10 percent. Should the firm purchase the piece of art?

Figure 4.3 **Cash Flows for Investment in Painting**



Our first thought might be to discount at the interest rate, yielding:

$$\frac{\$480{,}000}{1.10} = \$436{,}364$$

Because $436,364 is greater than $400,000, it looks at first glance as if the painting should be purchased. However, 10 percent is the return one can earn on a riskless investment. Because the painting is quite risky, a higher discount rate is called for. The manager chooses a rate of 25 percent to reflect this risk. In other words, he argues that a 25 percent expected return is fair compensation for an investment as risky as this painting.

The present value of the painting becomes:

$$\frac{\$480{,}000}{1.25} = \$384{,}000$$

Thus, the manager believes that the painting is currently overpriced at $400,000 and does not make the purchase.

The preceding analysis is typical of decision making in today's corporations, though real-world examples are, of course, much more complex. Unfortunately, any example with risk poses a problem not presented by a riskless example. In an example with riskless cash flows, the appropriate interest rate can be determined by simply checking with a few banks. The selection of the discount rate for a risky investment is quite a difficult task. We simply don't know at this point whether the discount rate on the painting in Example 4.2 should be 11 percent, 25 percent, 52 percent, or some other percentage.

Because the choice of a discount rate is so difficult, we merely wanted to broach the subject here. We must wait until the specific material on risk and return is covered in later chapters before a risk-adjusted analysis can be presented.

4.2 The Multiperiod Case

The previous section presented the calculation of future value and present value for one period only. We will now perform the calculations for the multiperiod case.

Future Value and Compounding

Suppose an individual were to make a loan of \$1. At the end of the first year, the borrower would owe the lender the principal amount of \$1 plus the interest on the loan at the interest rate of r. For the specific case where the interest rate is, say, 9 percent, the borrower owes the lender:

$$\$1 \times (1 + r) = \$1 \times 1.09 = \$1.09$$

At the end of the year, though, the lender has two choices. She can either take the \$1.09—or, more generally, $(1 + r)$—out of the financial market, or she can leave it in and lend it again for a second year. The process of leaving the money in the financial market and lending it for another year is called **compounding**.

Suppose the lender decides to compound her loan for another year. She does this by taking the proceeds from her first one-year loan, \$1.09, and lending this amount for the next year. At the end of next year, then, the borrower will owe her:

$$\$1 \times (1 + r) \times (1 + r) = \$1 \times (1 + r)^2 = 1 + 2r + r^2$$
$$\$1 \times (1.09) \times (1.09) = \$1 \times (1.09)^2 = \$1 + \$0.18 + \$0.0081 = \$1.1881$$

This is the total she will receive two years from now by compounding the loan.

In other words, the capital market enables the investor, by providing a ready opportunity for lending, to transform \$1 today into \$1.1881 at the end of two years. At the end of three years, the cash will be $\$1 \times (1.09)^3 = \$1.2950$.

The most important point to notice is that the total amount the lender receives is not just the \$1 that she lent plus two years' worth of interest on \$1:

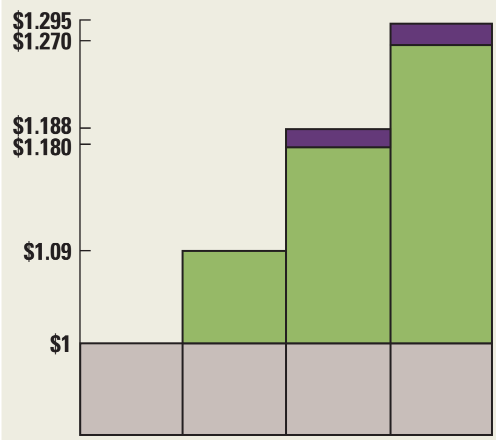
$$2 \times r = 2 \times \$0.09 = \$0.18$$

The lender also gets back an amount r^2, which is the interest in the second year on the interest that was earned in the first year. The term $2 \times r$ represents **simple interest** over the two years, and the term r^2 is referred to as the *interest on interest*. In our example, this latter amount is exactly:

$$r^2 = (\$0.09)^2 = \$0.0081$$

When cash is invested at **compound interest**, each interest payment is reinvested. With simple interest, the interest is not reinvested. Benjamin Franklin's statement, "Money makes money and the money that money makes makes more money," is a colorful way of explaining compound interest. The difference between compound interest and simple interest is illustrated in Figure 4.4. In this example, the difference does not amount to much because the loan is for \$1. If the loan were for \$1 million, the lender would receive \$1,188,100 in two years' time. Of this amount, \$8,100 is interest on interest. The lesson is that those small numbers beyond the decimal point can add up to big dollar amounts when the transactions are for big amounts. In addition, the longer-lasting the loan, the more important interest on interest becomes.

Figure 4.4
Simple and Compound Interest



The dark-shaded area indicates the difference between compound and simple interest. The difference is substantial over a period of many years or decades.

The general formula for an investment over many periods can be written as follows:

Future Value of an Investment:

$$\text{FV} = C_0 \times (1+r)^T \tag{4.3}$$

where C_0 is the cash to be invested at date 0 (i.e., today), r is the interest rate per period, and T is the number of periods over which the cash is invested.

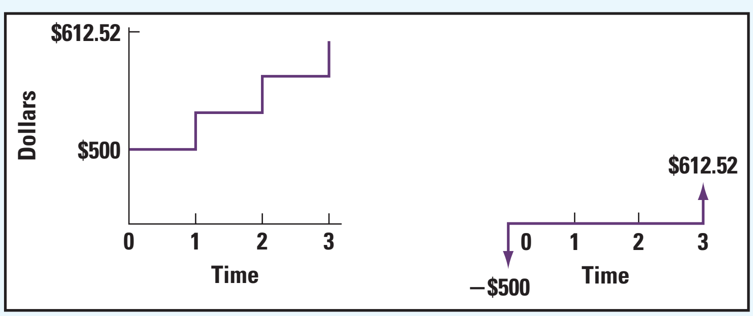
EXAMPLE 4.3

Interest on Interest Suh-Pyng Ku has put \$500 in a savings account at the First National Bank of Kent. The account earns 7 percent, compounded annually. How much will Ms. Ku have at the end of three years? The answer is:

$$\$500 \times 1.07 \times 1.07 \times 1.07 = \$500 \times (1.07)^3 = \$612.52$$

Figure 4.5 illustrates the growth of Ms. Ku's account.

Figure 4.5 **Suh-Pyng Ku's Savings Account**



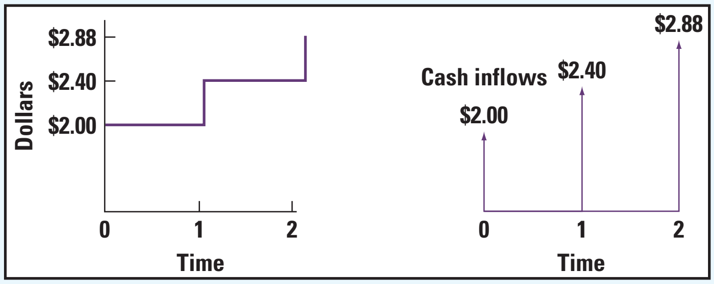
EXAMPLE 4.4

Compound Growth Jay Ritter invested $1,000 in the stock of the SDH Company. The company pays a current dividend of $2, which is expected to grow by 20 percent per year for the next two years. What will the dividend of the SDH Company be after two years? A simple calculation gives:

$$\$2 \times (1.20)^2 = \$2.88$$

Figure 4.6 illustrates the increasing value of SDH's dividends.

Figure 4.6 **The Growth of the SDH Dividends**



The two previous examples can be calculated in any one of several ways. The computations could be done by hand, by calculator, by spreadsheet, or with the help of a table. The appropriate table is Table A.3, which appears in the back of the text. This table presents *future value of $1 at the end of T periods*. The table is used by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Suh-Pyng Ku would look at the following portion of Table A.3:

	Interest Rate		
Period	**6%**	**7%**	**8%**
1	1.0600	1.0700	1.0800
2	1.1236	1.1449	1.1664
3	1.1910	1.2250	1.2597
4	1.2625	1.3108	1.3605

She could calculate the future value of her $500 as

$$\underset{\text{Initial investment}}{\$500} \times \underset{\text{Future value of \$1}}{1.2250} = \$612.50$$

In the example concerning Suh-Pyng Ku, we gave you both the initial investment and the interest rate and then asked you to calculate the future value. Alternatively, the interest rate could have been unknown, as shown in the following example.

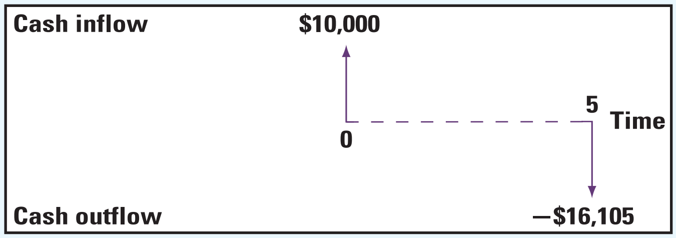
EXAMPLE 4.5

Finding the Rate Carl Voigt, who recently won $10,000 in the lottery, wants to buy a car in five years. Carl estimates that the car will cost $16,105 at that time. His cash flows are displayed in Figure 4.7.

What interest rate must he earn to be able to afford the car?

(*continued*)

Figure 4.7 **Cash Flows for Purchase of Carl Voigt's Car**



The ratio of purchase price to initial cash is:

$$\frac{\$16{,}105}{\$10{,}000} = 1.6105$$

Thus, he must earn an interest rate that allows $1 to become $1.6105 in five years. Table A.3 tells us that an interest rate of 10 percent will allow him to purchase the car.

We can express the problem algebraically as:

$$\$10{,}000 \times (1 + r)^5 = \$16{,}105$$

where r is the interest rate needed to purchase the car. Because $16,105/$10,000 = 1.6105, we have:

$$(1 + r)^5 = 1.6105$$
$$r = 10\%$$

Either the table or a hand calculator lets us solve for r.

The Power of Compounding: A Digression

Most people who have had any experience with compounding are impressed with its power over long periods. Take the stock market, for example. Ibbotson and Sinquefield have calculated what the stock market returned as a whole from 1926 through 2005.[1] They find that one dollar placed in these stocks at the beginning of 1926 would have been worth $2,657.56 at the end of 2005. This is 10.36 percent compounded annually for 80 years—that is, $(1.1036)^{80}$ = $2,657.56, ignoring a small rounding error.

The example illustrates the great difference between compound and simple interest. At 10.36 percent, simple interest on $1 is 10.36 cents a year. Simple interest over 80 years is $8.29 (=80 × $.1035). That is, an individual withdrawing 10.35 cents every year would have withdrawn $8.29 (=80 × $0.1035) over 80 years. This is quite a bit below the $2,657.56 that was obtained by reinvestment of all principal and interest.

The results are more impressive over even longer periods. A person with no experience in compounding might think that the value of $1 at the end of 160 years would be twice the value of $1 at the end of 80 years, if the yearly rate of return stayed the same. Actually the value of $1 at the end of 160 years would be the *square* of the value of $1 at the end of 80 years. That is, if the annual rate of return remained the same, a $1 investment in common stocks should be worth $7,062, 625.15 [=$1 × (2,657.56 × 2,657.56)].

A few years ago, an archaeologist unearthed a relic stating that Julius Caesar lent the Roman equivalent of one penny to someone. Because there was no record of the penny ever being repaid, the archaeologist wondered what the interest and principal would be if a

[1] *Stocks, Bonds, Bills, and Inflation* [SBBI]. 2006 Yearbook. Ibbotson Associates, Chicago, 2006.

descendant of Caesar tried to collect from a descendant of the borrower in the 20th century. The archaeologist felt that a rate of 6 percent might be appropriate. To his surprise, the principal and interest due after more than 2,000 years was vastly greater than the entire wealth on earth.

The power of compounding can explain why the parents of well-to-do families frequently bequeath wealth to their grandchildren rather than to their children. That is, they skip a generation. The parents would rather make the grandchildren very rich than make the children moderately rich. We have found that in these families the grandchildren have a more positive view of the power of compounding than do the children.

EXAMPLE 4.6

How Much for That Island? Some people have said that it was the best real estate deal in history. Peter Minuit, director general of New Netherlands, the Dutch West India Company's colony in North America, in 1626 allegedly bought Manhattan Island for 60 guilders' worth of trinkets from native Americans. By 1667, the Dutch were forced by the British to exchange it for Suriname (perhaps the worst real estate deal ever). This sounds cheap; but did the Dutch really get the better end of the deal? It is reported that 60 guilders was worth about $24 at the prevailing exchange rate. If the native Americans had sold the trinkets at a fair market value and invested the $24 at 5 percent (tax free), it would now, about 380 years later, be worth more than $2.5 billion. Today, Manhattan is undoubtedly worth more than $2.5 billion, so at a 5 percent rate of return the native Americans got the worst of the deal. However, if invested at 10 percent, the amount of money they received would be worth about:

$$\$24(1 + r)^T = 24 \times 1.1^{380} \cong \$129 \text{ quadrillion}$$

This is a lot of money. In fact, $129 quadrillion is more than all the real estate in the world is worth today. Note that no one in the history of the world has ever been able to find an investment yielding 10 percent every year for 380 years.

Present Value and Discounting

We now know that an annual interest rate of 9 percent enables the investor to transform $1 today into $1.1881 two years from now. In addition, we would like to know the following:

> How much would an investor need to lend today so that she could receive $1 two years from today?

Algebraically, we can write this as:

$$\text{PV} \times (1.09)^2 = \$1$$

In the preceding equation, PV stands for present value, the amount of money we must lend today to receive $1 in two years' time.

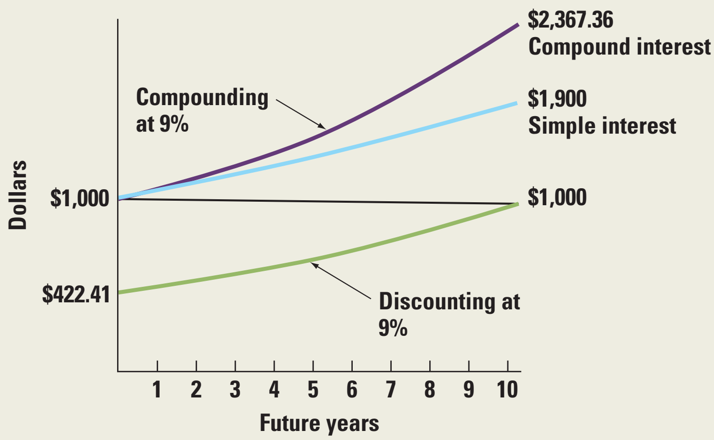
Solving for PV in this equation, we have:

$$\text{PV} = \frac{\$1}{1.1881} = \$.84$$

This process of calculating the present value of a future cash flow is called **discounting**. It is the opposite of compounding. The difference between compounding and discounting is illustrated in Figure 4.8.

To be certain that $.84 is in fact the present value of $1 to be received in two years, we must check whether or not, if we lent $.84 today and rolled over the loan for two years, we would get exactly $1 back. If this were the case, the capital markets would be saying that $1

Figure 4.8
Compounding and Discounting



The top line shows the growth of $1,000 at compound interest with the funds invested at 9 percent: $\$1{,}000 \times (1.09)^{10} = \$2{,}367.36$. Simple interest is shown on the next line. It is $\$1{,}000 + [10 \times (\$1{,}000 \times .09)] = \$1{,}900$. The bottom line shows the discounted value of $1,000 if the interest rate is 9 percent.

received in two years' time is equivalent to having $.84 today. Checking the exact numbers, we get:

$$\$.84168 \times 1.09 \times 1.09 = \$1$$

In other words, when we have capital markets with a sure interest rate of 9 percent, we are indifferent between receiving $.84 today or $1 in two years. We have no reason to treat these two choices differently from each other because if we had $.84 today and lent it out for two years, it would return $1 to us at the end of that time. The value .84 $[=1/(1.09)^2]$ is called the **present value factor**. It is the factor used to calculate the present value of a future cash flow.

In the multiperiod case, the formula for PV can be written as follows:

Present Value of Investment:

$$\text{PV} = \frac{C_T}{(1+r)^T} \tag{4.4}$$

Here, C_T is the cash flow at date T and r is the appropriate discount rate.

EXAMPLE 4.7

Multiperiod Discounting Bernard Dumas will receive $10,000 three years from now. Bernard can earn 8 percent on his investments, so the appropriate discount rate is 8 percent. What is the present value of his future cash flow? The answer is:

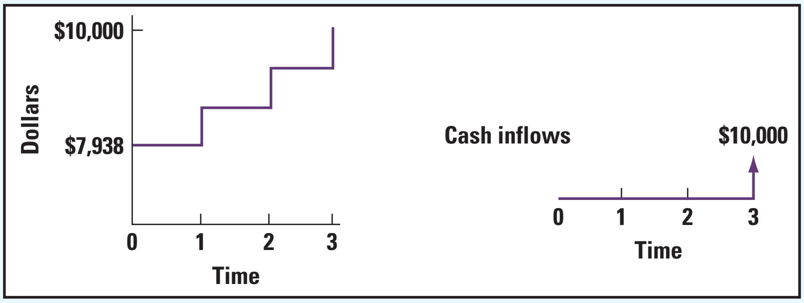
$$\begin{aligned} \text{PV} &= \$10{,}000 \times \left(\frac{1}{1.08}\right)^3 \\ &= \$10{,}000 \times .7938 \\ &= \$7{,}938 \end{aligned}$$

Figure 4.9 illustrates the application of the present value factor to Bernard's investment.

When his investments grow at an 8 percent rate of interest, Bernard Dumas is equally inclined toward receiving $7,938 now and receiving $10,000 in three years' time. After all, he could convert the $7,938 he receives today into $10,000 in three years by lending it at an interest rate of 8 percent.

(continued)

Figure 4.9 **Discounting Bernard Dumas's Opportunity**



Bernard Dumas could have reached his present value calculation in one of several ways. The computation could have been done by hand, by calculator, with a spreadsheet, or with the help of Table A.1, which appears in the back of the text. This table presents the *present value of $1 to be received after T periods*. We use the table by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Bernard Dumas would look at the following portion of Table A.1:

	Interest Rate		
Period	**7%**	**8%**	**9%**
1	.9346	.9259	.9174
2	.8734	.8573	.8417
3	.8163	$\boxed{.7938}$	.7722
4	.7629	.7350	.7084

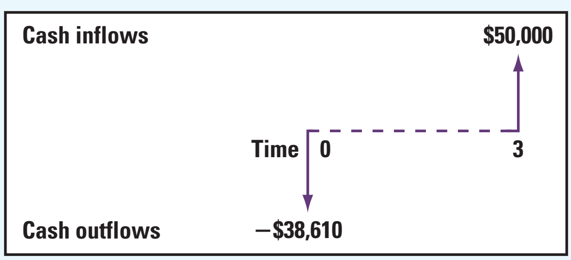
The appropriate present value factor is .7938.

In the preceding example we gave both the interest rate and the future cash flow. Alternatively, the interest rate could have been unknown.

EXAMPLE 4.8

Finding the Rate A customer of the Chaffkin Corp. wants to buy a tugboat today. Rather than paying immediately, he will pay $50,000 in three years. It will cost the Chaffkin Corp. $38,610 to build the tugboat immediately. The relevant cash flows to Chaffkin Corp. are displayed in Figure 4.10. By charging what interest rate would the Chaffkin Corp. neither gain nor lose on the sale?

Figure 4.10 **Cash Flows for Tugboat**



(*continued*)

The ratio of construction cost (present value) to sale price (future value) is:

$$\frac{\$38{,}610}{\$50{,}000} = 0.7722$$

We must determine the interest rate that allows \$1 to be received in three years to have a present value of \$.7722. Table A.1 tells us that 9 percent is that interest rate.

Frequently, an investor or a business will receive more than one cash flow. The present value of the set of cash flows is simply the sum of the present values of the individual cash flows. This is illustrated in the following example.

EXAMPLE 4.9

Cash Flow Valuation Dennis Draper has won the Kentucky State Lottery and will receive the following set of cash flows over the next two years:

Year	Cash Flow
1	\$2,000
2	\$5,000

Mr. Draper can currently earn 6 percent in his money market account, so the appropriate discount rate is 6 percent. The present value of the cash flows is:

Year	Cash Flow × Present Value Factor	= Present Value
1	$\$2{,}000 \times \frac{1}{1.06} = \$2{,}000 \times .943$	$= \$1{,}887$
2	$\$5{,}000 \times \left(\frac{1}{1.06}\right)^2 = \$5{,}000 \times .890$	$= \$4{,}450$
	Total	\$6,337

In other words, Mr. Draper is equally inclined toward receiving \$6,337 today and receiving \$2,000 and \$5,000 over the next two years.

EXAMPLE 4.10

NPV Finance.com has an opportunity to invest in a new high-speed computer that costs \$50,000. The computer will generate cash flows (from cost savings) of \$25,000 one year from now, \$20,000 two years from now, and \$15,000 three years from now. The computer will be worthless after three years, and no additional cash flows will occur. Finance.com has determined that the appropriate discount rate is 7 percent for this investment. Should Finance.com make this investment in a new high-speed computer? What is the net present value of the investment?

(*continued*)

The cash flows and present value factors of the proposed computer are as follows:

	Cash Flows	Present Value Factor
Year 0	$-$$50,000	$1 = 1$
1	$25,000	$\frac{1}{1.07} = .9346$
2	$20,000	$\left(\frac{1}{1.07}\right)^2 = .8734$
3	$15,000	$\left(\frac{1}{1.07}\right)^3 = .8163$

The present value of the cash flows is:

Cash flows $\times$ Present value factor $=$ Present value

Year 0	$-$$50,000 $\times$ 1	=	$-$$50,000
1	$25,000 $\times$ .9346	=	$23,365
2	$20,000 $\times$ .8734	=	$17,468
3	$15,000 $\times$ .8163	=	$12,244.5
		Total:	$ 3,077.5

Finance.com should invest in the new high-speed computer because the present value of its future cash flows is greater than its cost. The NPV is $3,077.5.

The Algebraic Formula

To derive an algebraic formula for the net present value of a cash flow, recall that the PV of receiving a cash flow one year from now is:

$$\text{PV} = C_1/(1 + r)$$

and the PV of receiving a cash flow two years from now is:

$$\text{PV} = C_2/(1 + r)^2$$

We can write the NPV of a T-period project as:

$$\text{NPV} = -C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = -C_0 + \sum_{i=1}^{T} \frac{C_i}{(1+r)^i} \tag{4.5}$$

The initial flow, $-C_0$, is assumed to be negative because it represents an investment. The Σ is shorthand for the sum of the series.

We will close out this section by answering the question we posed at the beginning of the chapter concerning baseball player A.J. Burnett's contract. Remember that the contract called for a signing bonus of $6 million to be paid immediately, plus salary and bonuses of $49 million to be distributed as $1 million in 2006 and $12 million per year for

2007 through 2010. If 12 percent is the appropriate interest rate, what kind of deal did the Toronto Blue Jays pitch to A.J.?

To answer, we can calculate the present value by discounting each year's salary back to the present as follows (notice we assumed the future salaries will be paid at the end of the year):

$$
\begin{array}{lll}
\text{Year 0:} & \$\ \ 6{,}000{,}000 & = \$6{,}000{,}000 \\
\text{Year 1:} & \$\ \ 1{,}000{,}000 \times 1/1.12^1 & = \$\ \ \ 892{,}857.14 \\
\text{Year 2:} & \$12{,}000{,}000 \times 1/1.12^2 & = \$9{,}566{,}326.53 \\
\vdots & \vdots & \vdots \\
\text{Year 5:} & \$12{,}000{,}000 \times 1/1.12^5 & = \$6{,}809{,}122.27
\end{array}
$$

If you fill in the missing rows and then add (do it for practice), you will see that Burnett's contract had a present value of about \$39.44 million, or only about 70 percent of the \$55 million reported value, but still pretty good. And of course, playing for the Toronto Blue Jays, Burnett will probably have his Octobers free as well.

4.3 Compounding Periods

So far, we have assumed that compounding and discounting occur yearly. Sometimes, compounding may occur more frequently than just once a year. For example, imagine that a bank pays a 10 percent interest rate "compounded semiannually." This means that a \$1,000 deposit in the bank would be worth \$1,000 × 1.05 = \$1,050 after six months, and \$1,050 × 1.05 = \$1,102.50 at the end of the year.

The end-of-the-year wealth can be written as:

$$\$1{,}000\left(1+\frac{.10}{2}\right)^2 = \$1{,}000 \times (1.05)^2 = \$1{,}102.50$$

Of course, a \$1,000 deposit would be worth \$1,100 (\$1,000 × 1.10) with yearly compounding. Note that the future value at the end of one year is greater with semiannual compounding than with yearly compounding. With yearly compounding, the original \$1,000 remains the investment base for the full year. The original \$1,000 is the investment base only for the first six months with semiannual compounding. The base over the second six months is \$1,050. Hence one gets *interest on interest* with semiannual compounding.

Because \$1,000 × 1.1025 = \$1,102.50, 10 percent compounded semiannually is the same as 10.25 percent compounded annually. In other words, a rational investor could not care less whether she is quoted a rate of 10 percent compounded semiannually or a rate of 10.25 percent compounded annually.

Quarterly compounding at 10 percent yields wealth at the end of one year of:

$$\$1{,}000\left(1+\frac{.10}{4}\right)^4 = \$1{,}103.81$$

More generally, compounding an investment m times a year provides end-of-year wealth of:

$$C_0\left(1+\frac{r}{m}\right)^m \tag{4.6}$$

where C_0 is the initial investment and r is the **stated annual interest rate**. The stated annual interest rate is the annual interest rate without consideration of compounding. Banks and other financial institutions may use other names for the stated annual interest rate. **Annual percentage rate (APR)** is perhaps the most common synonym.

EXAMPLE 4.11

EARs What is the end-of-year wealth if Jane Christine receives a stated annual interest rate of 24 percent compounded monthly on a $1 investment?

Using Equation 4.6, her wealth is:

$$\$1\left(1+\frac{.24}{12}\right)^{12} = \$1 \times (1.02)^{12}$$
$$= \$1.2682$$

The annual rate of return is 26.82 percent. This annual rate of return is called either the **effective annual rate (EAR)** or the **effective annual yield (EAY)**. Due to compounding, the effective annual interest rate is greater than the stated annual interest rate of 24 percent. Algebraically, we can rewrite the effective annual interest rate as follows:

Effective Annual Rate:

$$\left(1+\frac{r}{m}\right)^{m} - 1 \qquad \textbf{(4.7)}$$

Students are often bothered by the subtraction of 1 in Equation 4.7. Note that end-of-year wealth is composed of both the interest earned over the year and the original principal. We remove the original principal by subtracting 1 in Equation 4.7.

EXAMPLE 4.12

Compounding Frequencies If the stated annual rate of interest, 8 percent, is compounded quarterly, what is the effective annual rate?

Using Equation 4.7, we have:

$$\left(1+\frac{r}{m}\right)^{m} - 1 = \left(1+\frac{.08}{4}\right)^{4} - 1 = .0824 = 8.24\%$$

Referring back to our original example where $C_0 = \$1{,}000$ and $r = 10\%$, we can generate the following table:

C_0	Compounding frequency (m)	C_1	Effective Annual Rate = $\left(1+\frac{r}{m}\right)^{m} - 1$
$1,000	Yearly ($m = 1$)	$1,100.00	.10
1,000	Semiannually ($m = 2$)	1,102.50	.1025
1,000	Quarterly ($m = 4$)	1,103.81	.10381
1,000	Daily ($m = 365$)	1,105.16	.10516

Distinction between Stated Annual Interest Rate and Effective Annual Rate

The distinction between the stated annual interest rate (SAIR), or APR, and the effective annual rate (EAR) is frequently troubling to students. We can reduce the confusion by noting that the SAIR becomes meaningful only if the compounding interval is given. For example, for an SAIR of 10 percent, the future value at the end of one year with semiannual compounding is $[1 + (.10/2)]^2 = 1.1025$. The future value with quarterly compounding is $[1 + (.10/4)]^4 = 1.1038$. If the SAIR is 10 percent but no compounding interval is given, we cannot calculate future value. In other words, we do not know whether to compound semiannually, quarterly, or over some other interval.

By contrast, the EAR is meaningful *without* a compounding interval. For example, an EAR of 10.25 percent means that a \$1 investment will be worth \$1.1025 in one year. We can think of this as an SAIR of 10 percent with semiannual compounding or an SAIR of 10.25 percent with annual compounding, or some other possibility.

There can be a big difference between an SAIR and an EAR when interest rates are large. For example, consider "payday loans." Payday loans are short-term loans made to consumers, often for less than two weeks, and are offered by companies such as AmeriCash Advance and National Payday. The loans work like this: You write a check today that is postdated. When the check date arrives, you go to the store and pay the cash for the check, or the company cashes the check. For example, AmeriCash Advance allows you to write a postdated check for \$125 for 15 days later. In this case, they would give you \$100 today. So, what are the APR and EAR of this arrangement? First, we need to find the interest rate, which we can find by the FV equation as follows:

$$\begin{aligned} \text{FV} &= \text{PV}\,(1 + r)^T \\ \$125 &= \$100 \times (1 + r)^1 \\ 1.25 &= (1 + r) \\ r &= .25 \text{ or } 25\% \end{aligned}$$

That doesn't seem too bad until you remember this is the interest rate for *15 days*! The APR of the loan is:

$$\begin{aligned} \text{APR} &= .25 \times 365/15 \\ \text{APR} &= 6.0833 \text{ or } 608.33\% \end{aligned}$$

And the EAR for this loan is:

$$\begin{aligned} \text{EAR} &= (1 + r/m)^m - 1 \\ \text{EAR} &= (1 + .25)^{365/15} - 1 \\ \text{EAR} &= 227.1096 \text{ or } 22{,}710.96\% \end{aligned}$$

Now that's an interest rate! Just to see what a difference a day (or three) makes, let's look at National Payday's terms. This company will allow you to write a postdated check for the same amount, but will allow you 18 days to repay. Check for yourself that the APR of this arrangement is 506.94 percent and the EAR is 9,128.26 percent. This is lower, but still not a loan we recommend.

Compounding over Many Years

Equation 4.6 applies for an investment over one year. For an investment over one or more (T) years, the formula becomes this:

Future Value with Compounding:

$$\text{FV} = C_0\left(1 + \frac{r}{m}\right)^{mT} \qquad \textbf{(4.8)}$$

EXAMPLE 4.13

Multiyear Compounding Harry DeAngelo is investing $5,000 at a stated annual interest rate of 12 percent per year, compounded quarterly, for five years. What is his wealth at the end of five years?

Using Equation 4.8, his wealth is:

$$\$5{,}000 \times \left(1 + \frac{.12}{4}\right)^{4\times5} = \$5{,}000 \times (1.03)^{20} = \$5{,}000 \times 1.8061 = \$9{,}030.50$$

Continuous Compounding

The previous discussion shows that we can compound much more frequently than once a year. We could compound semiannually, quarterly, monthly, daily, hourly, each minute, or even more often. The limiting case would be to compound every infinitesimal instant, which is commonly called **continuous compounding**. Surprisingly, banks and other financial institutions sometimes quote continuously compounded rates, which is why we study them.

Though the idea of compounding this rapidly may boggle the mind, a simple formula is involved. With continuous compounding, the value at the end of T years is expressed as:

$$C_0 \times e^{rT} \tag{4.9}$$

where C_0 is the initial investment, r is the stated annual interest rate, and T is the number of years over which the investment runs. The number e is a constant and is approximately equal to 2.718. It is not an unknown like C_0, r, and T.

EXAMPLE 4.14

Continuous Compounding Linda DeFond invested $1,000 at a continuously compounded rate of 10 percent for one year. What is the value of her wealth at the end of one year?

From Equation 4.9 we have:

$$\$1{,}000 \times e^{0.10} = \$1{,}000 \times 1.1052 = \$1{,}105.20$$

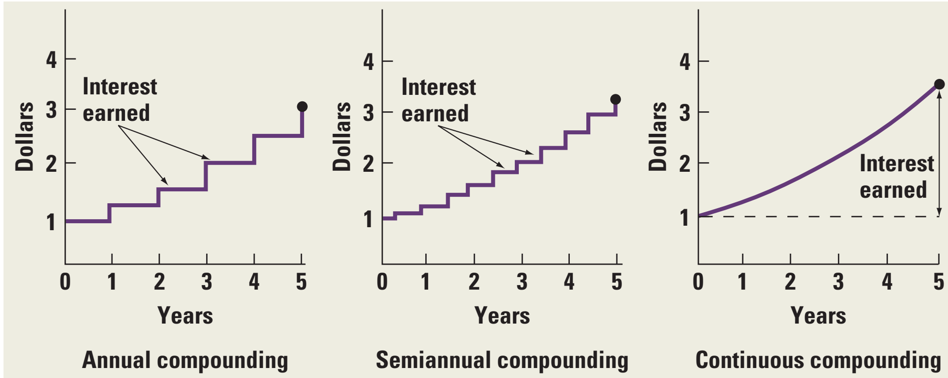
This number can easily be read from Table A.5. We merely set r, the value on the horizontal dimension, to 10 percent and T, the value on the vertical dimension, to 1. For this problem the relevant portion of the table is shown here:

Period	Continuously Compounded Rate (r)		
(T)	9%	10%	11%
1	1.0942	1.1052	1.1163
2	1.1972	1.2214	1.2461
3	1.3100	1.3499	1.3910

Note that a continuously compounded rate of 10 percent is equivalent to an annually compounded rate of 10.52 percent. In other words, Linda DeFond would not care whether her bank quoted a continuously compounded rate of 10 percent or a 10.52 percent rate, compounded annually.

Figure 4.11

Annual, Semiannual, and Continuous Compounding



EXAMPLE 4.15

Continuous Compounding, Continued Linda DeFond's brother, Mark, invested $1,000 at a continuously compounded rate of 10 percent for two years.

The appropriate formula here is:

$$\$1{,}000 \times e^{.10 \times 2} = \$1{,}000 \times e^{.20} = \$1{,}221.40$$

Using the portion of the table of continuously compounded rates shown in the previous example, we find the value to be 1.2214.

Figure 4.11 illustrates the relationship among annual, semiannual, and continuous compounding. Semiannual compounding gives rise to both a smoother curve and a higher ending value than does annual compounding. Continuous compounding has both the smoothest curve and the highest ending value of all.

EXAMPLE 4.16

Present Value with Continuous Compounding The Michigan state lottery is going to pay you $1,000 at the end of four years. If the annual continuously compounded rate of interest is 8 percent, what is the present value of this payment?

$$\$1{,}000 \times \frac{1}{e^{.08 \times 4}} = \$1{,}000 \times \frac{1}{1.3771} = \$726.16$$

4.4 Simplifications

The first part of this chapter has examined the concepts of future value and present value. Although these concepts allow us to answer a host of problems concerning the time value of money, the human effort involved can be excessive. For example, consider a bank calculating the present value of a 20-year monthly mortgage. This mortgage has 240 (=20 × 12) payments, so a lot of time is needed to perform a conceptually simple task.

Because many basic finance problems are potentially time-consuming, we search for simplifications in this section. We provide simplifying formulas for four classes of cash flow streams:

- Perpetuity.
- Growing perpetuity.

- Annuity.
- Growing annuity.

Perpetuity

A **perpetuity** is a constant stream of cash flows without end. If you are thinking that perpetuities have no relevance to reality, it will surprise you that there is a well-known case of an unending cash flow stream: the British bonds called *consols*. An investor purchasing a consol is entitled to receive yearly interest from the British government forever.

How can the price of a consol be determined? Consider a consol that pays a coupon of C dollars each year and will do so forever. Simply applying the PV formula gives us:

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

where the dots at the end of the formula stand for the infinite string of terms that continues the formula. Series like the preceding one are called *geometric series*. It is well known that even though they have an infinite number of terms, the whole series has a finite sum because each term is only a fraction of the preceding term. Before turning to our calculus books, though, it is worth going back to our original principles to see if a bit of financial intuition can help us find the PV.

The present value of the consol is the present value of all of its future coupons. In other words, it is an amount of money that, if an investor had it today, would enable him to achieve the same pattern of expenditures that the consol and its coupons would. Suppose an investor wanted to spend exactly C dollars each year. If he had the consol, he could do this. How much money must he have today to spend the same amount? Clearly, he would need exactly enough so that the interest on the money would be C dollars per year. If he had any more, he could spend more than C dollars each year. If he had any less, he would eventually run out of money spending C dollars per year.

The amount that will give the investor C dollars each year, and therefore the present value of the consol, is simply:

$$\text{PV} = \frac{C}{r} \tag{4.10}$$

To confirm that this is the right answer, notice that if we lend the amount C/r, the interest it earns each year will be:

$$\text{Interest} = \frac{C}{r} \times r = C$$

which is exactly the consol payment. We have arrived at this formula for a consol:

Formula for Present Value of Perpetuity:

$$\begin{aligned}\text{PV} &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \\ &= \frac{C}{r}\end{aligned} \tag{4.11}$$

It is comforting to know how easily we can use a bit of financial intuition to solve this mathematical problem.

EXAMPLE 4.17

Perpetuities Consider a perpetuity paying \$100 a year. If the relevant interest rate is 8 percent, what is the value of the consol?

Using Equation 4.10 we have:

$$\text{PV} = \frac{\$100}{.08} = \$1{,}250$$

Now suppose that interest rates fall to 6 percent. Using Equation 4.10 the value of the perpetuity is:

$$\text{PV} = \frac{\$100}{.06} = \$1{,}666.67$$

Note that the value of the perpetuity rises with a drop in the interest rate. Conversely, the value of the perpetuity falls with a rise in the interest rate.

Growing Perpetuity

Imagine an apartment building where cash flows to the landlord after expenses will be \$100,000 next year. These cash flows are expected to rise at 5 percent per year. If one assumes that this rise will continue indefinitely, the cash flow stream is termed a **growing perpetuity**. The relevant interest rate is 11 percent. Therefore, the appropriate discount rate is 11 percent, and the present value of the cash flows can be represented as:

$$\text{PV} = \frac{\$100{,}000}{1.11} + \frac{\$100{,}000(1.05)}{(1.11)^2} + \frac{\$100{,}000(1.05)^2}{(1.11)^3} + \cdots + \frac{\$100{,}000(1.05)^{N-1}}{(1.11)^N} + \cdots$$

Algebraically, we can write the formula as:

$$\text{PV} = \frac{C}{1+r} + \frac{C \times (1+g)}{(1+r)^2} + \frac{C \times (1+g)^2}{(1+r)^3} + \cdots + \frac{C \times (1+g)^{N-1}}{(1+r)^N} + \cdots$$

where C is the cash flow to be received one period hence, g is the rate of growth per period, expressed as a percentage, and r is the appropriate discount rate.

Fortunately, this formula reduces to the following simplification:

Formula for Present Value of Growing Perpetuity:

$$\text{PV} = \frac{C}{r-g} \tag{4.12}$$

From Equation 4.12 the present value of the cash flows from the apartment building is:

$$\frac{\$100{,}000}{.11 - .05} = \$1{,}666{,}667$$

There are three important points concerning the growing perpetuity formula:

1. *The numerator*: The numerator in Equation 4.12 is the cash flow one period hence, not at date 0. Consider the following example.

EXAMPLE 4.18

Paying Dividends Rothstein Corporation is *just about* to pay a dividend of \$3.00 per share. Investors anticipate that the annual dividend will rise by 6 percent a year forever. The applicable discount rate is 11 percent. What is the price of the stock today?

The numerator in Equation 4.12 is the cash flow to be received next period. Since the growth rate is 6 percent, the dividend next year is \$3.18 (=\$3.00 × 1.06). The price of the stock today is:

$$\underset{}{\$66.60} = \underset{\substack{\text{Imminent}\\\text{dividend}}}{\$3.00} + \underset{\substack{\text{Present value of all}\\\text{dividends beginning}\\\text{a year from now}}}{\frac{\$3.18}{.11 - .06}}$$

The price of \$66.60 includes both the dividend to be received immediately and the present value of all dividends beginning a year from now. Equation 4.12 makes it possible to calculate only the present value of all dividends beginning a year from now. Be sure you understand this example; test questions on this subject always seem to trip up a few of our students.

2. *The discount rate and the growth rate*: The discount rate r must be greater than the growth rate g for the growing perpetuity formula to work. Consider the case in which the growth rate approaches the interest rate in magnitude. Then, the denominator in the growing perpetuity formula gets infinitesimally small and the present value grows infinitely large. The present value is in fact undefined when r is less than g.
3. *The timing assumption*: Cash generally flows into and out of real-world firms both randomly and nearly continuously. However, Equation 4.12 assumes that cash flows are received and disbursed at regular and discrete points in time. In the example of the apartment, we assumed that the net cash flows of \$100,000 occurred only once a year. In reality, rent checks are commonly received every month. Payments for maintenance and other expenses may occur anytime within the year.

   We can apply the growing perpetuity formula of Equation 4.12 only by assuming a regular and discrete pattern of cash flow. Although this assumption is sensible because the formula saves so much time, the user should never forget that it is an *assumption*. This point will be mentioned again in the chapters ahead.

A few words should be said about terminology. Authors of financial textbooks generally use one of two conventions to refer to time. A minority of financial writers treat cash flows as being received on exact *dates*—for example date 0, date 1, and so forth. Under this convention, date 0 represents the present time. However, because a year is an interval, not a specific moment in time, the great majority of authors refer to cash flows that occur at the end of a year (or alternatively, the end of a *period*). Under this *end-of-the-year* convention, the end of year 0 is the present, the end of year 1 occurs one period hence, and so on. (The beginning of year 0 has already passed and is not generally referred to.)[2]

The interchangeability of the two conventions can be seen from the following chart:

Date 0 = Now	Date 1	Date 2	Date 3	. . .
End of year 0 = Now	End of year 1	End of year 2	End of year 3	. . .

[2]Sometimes, financial writers merely speak of a cash flow in year x. Although this terminology is ambiguous, such writers generally mean the *end of year x*.

We strongly believe that the *dates convention* reduces ambiguity. However, we use both conventions because you are likely to see the *end-of-year convention* in later courses. In fact, both conventions may appear in the same example for the sake of practice.

Annuity

An **annuity** is a level stream of regular payments that lasts for a fixed number of periods. Not surprisingly, annuities are among the most common kinds of financial instruments. The pensions that people receive when they retire are often in the form of an annuity. Leases and mortgages are also often annuities.

To figure out the present value of an annuity we need to evaluate the following equation:

$$\frac{C}{1+r}+\frac{C}{(1+r)^2}+\frac{C}{(1+r)^3}+\cdots+\frac{C}{(1+r)^T}$$

The present value of receiving the coupons for only T periods must be less than the present value of a consol, but how much less? To answer this, we have to look at consols a bit more closely.

Consider the following time chart:

	Now						
Date (or end of year)	0	1	2	3	T	$(T+1)$	$(T+2)$
Consol 1		C	C	$C\ldots$	C	C	$C\ldots$
Consol 2						C	$C\ldots$
Annuity		C	C	$C\ldots$	C		

Consol 1 is a normal consol with its first payment at date 1. The first payment of consol 2 occurs at date $T+1$.

The present value of having a cash flow of C at each of T dates is equal to the present value of consol 1 minus the present value of consol 2. The present value of consol 1 is given by:

$$\text{PV}=\frac{C}{r} \tag{4.13}$$

Consol 2 is just a consol with its first payment at date $T+1$. From the perpetuity formula, this consol will be worth C/r at date T.[3] However, we do not want the value at date T. We want the value now, in other words, the present value at date 0. We must discount C/r back by T periods. Therefore, the present value of consol 2 is:

$$\text{PV}=\frac{C}{r}\left[\frac{1}{(1+r)^T}\right] \tag{4.14}$$

The present value of having cash flows for T years is the present value of a consol with its first payment at date 1 minus the present value of a consol with its first payment at date

[3]Students frequently think that C/r is the present value at date $T+1$ because the consol's first payment is at date $T+1$. However, the formula values the consol as of one period prior to the first payment.

$\mathcal{T}$ + 1. Thus the present value of an annuity is Equation 4.13 minus Equation 4.14. This can be written as:

$$\frac{C}{r} - \frac{C}{r}\left[\frac{1}{(1+r)^{\mathcal{T}}}\right]$$

This simplifies to the following:

Formula for Present Value of Annuity:

$$\text{PV} = C\left[\frac{1}{r} - \frac{1}{r(1+r)^{\mathcal{T}}}\right]$$

This can also be written as:

$$\text{PV} = C\left[\frac{1 - \dfrac{1}{(1+r)^{\mathcal{T}}}}{r}\right] \tag{4.15}$$

EXAMPLE 4.19

Lottery Valuation Mark Young has just won the state lottery, paying \$50,000 a year for 20 years. He is to receive his first payment a year from now. The state advertises this as the Million Dollar Lottery because \$1,000,000 = \$50,000 × 20. If the interest rate is 8 percent, what is the true value of the lottery?

Equation 4.15 yields:

$$\begin{array}{l} \text{Present value of} \\ \text{Million Dollar Lottery} \end{array} = \$50{,}000 \times \left[\frac{1 - \dfrac{1}{(1.08)^{20}}}{.08}\right]$$

$$\begin{array}{lll} \text{Periodic payment} & & \text{Annuity factor} \\ = \$50{,}000 & \times & 9.8181 \\ = \$490{,}905 & & \end{array}$$

Rather than being overjoyed at winning, Mr. Young sues the state for misrepresentation and fraud. His legal brief states that he was promised \$1 million but received only \$490,905.

The term we use to compute the present value of the stream of level payments, C, for $\mathcal{T}$ years is called an **annuity factor**. The annuity factor in the current example is 9.8181. Because the annuity factor is used so often in PV calculations, we have included it in Table A.2 in the back of this book. The table gives the values of these factors for a range of interest rates, r, and maturity dates, $\mathcal{T}$.

The annuity factor as expressed in the brackets of Equation 4.15 is a complex formula. For simplification, we may from time to time refer to the annuity factor as:

$$A_r^{\mathcal{T}}$$

This expression stands for the present value of \$1 a year for $\mathcal{T}$ years at an interest rate of r.

We can also provide a formula for the future value of an annuity:

$$\text{FV} = C\left[\frac{(1+r)^{\mathcal{T}}}{r} - \frac{1}{r}\right] = C\left[\frac{(1+r)^{\mathcal{T}} - 1}{r}\right] \tag{4.16}$$

As with present value factors for annuities, we have compiled future value factors in Table A.3 in the back of this book.

EXAMPLE 4.20

Retirement Investing Suppose you put $3,000 per year into a Roth IRA. The account pays 6 percent interest per year. How much will you have when you retire in 30 years?

This question asks for the future value of an annuity of $3,000 per year for 30 years at 6 percent, which we can calculate as follows:

$$\begin{aligned} FV = C\left[\frac{(1+r)^T - 1}{r}\right] &= \$3{,}000 \times \left[\frac{1.06^{30} - 1}{.06}\right] \\ &= \$3{,}000 \times 79.0582 \\ &= \$237{,}174.56 \end{aligned}$$

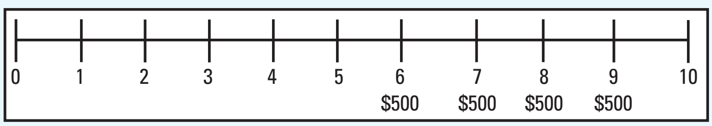
So, you'll have close to a quarter million dollars in the account.

Our experience is that annuity formulas are not hard, but tricky, for the beginning student. We present four tricks next.

Trick 1: A Delayed Annuity One of the tricks in working with annuities or perpetuities is getting the timing exactly right. This is particularly true when an annuity or perpetuity begins at a date many periods in the future. We have found that even the brightest beginning student can make errors here. Consider the following example.

EXAMPLE 4.21

Delayed Annuities Danielle Caravello will receive a four-year annuity of $500 per year, beginning at date 6. If the interest rate is 10 percent, what is the present value of her annuity? This situation can be graphed as follows:



The analysis involves two steps:

1. Calculate the present value of the annuity using Equation 4.15:

Present Value of Annuity at Date 5:

$$\begin{aligned} \$500\left[\frac{1 - \dfrac{1}{(1.10)^4}}{.10}\right] &= \$500 \times A^4_{.10} \\ &= \$500 \times 3.1699 \\ &= \$1{,}584.95 \end{aligned}$$

Note that $1,584.95 represents the present value at *date 5*.

Students frequently think that $1,584.95 is the present value at date 6 because the annuity begins at date 6. However, our formula values the annuity as of one period prior to the first payment. This can be seen in the most typical case where the first payment occurs at date 1. The formula values the annuity as of date 0 in that case.

2. Discount the present value of the annuity back to date 0:

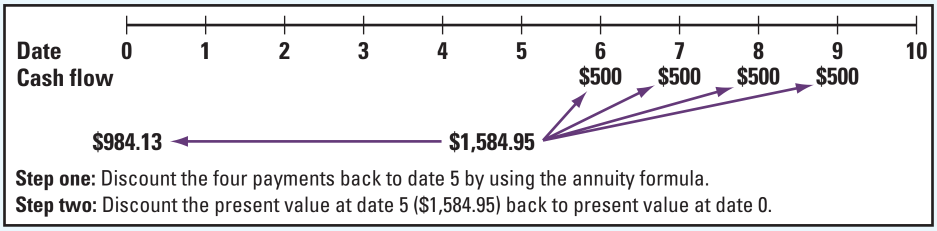
Present Value at Date 0:

$$\frac{\$1{,}584.95}{(1.10)^5} = \$984.13$$

(*continued*)

Again, it is worthwhile mentioning that because the annuity formula brings Danielle's annuity back to date 5, the second calculation must discount over the remaining five periods. The two-step procedure is graphed in Figure 4.12.

Figure 4.12 **Discounting Danielle Caravello's Annuity**



Trick 2: Annuity due The annuity formula of Equation 4.15 assumes that the first annuity payment begins a full period hence. This type of annuity is sometimes called an *annuity in arrears* or an *ordinary annuity*. What happens if the annuity begins today—in other words, at date 0?

EXAMPLE 4.22

Annuity Due In a previous example, Mark Young received $50,000 a year for 20 years from the state lottery. In that example, he was to receive the first payment a year from the winning date. Let us now assume that the first payment occurs immediately. The total number of payments remains 20.

Under this new assumption, we have a 19-date annuity with the first payment occurring at date 1—plus an extra payment at date 0. The present value is:

$$\underset{\text{Payment at date 0}}{\$50{,}000} \quad + \quad \underset{\text{19-year annuity}}{\$50{,}000 \times A^{19}_{.08}}$$

$$= \$50{,}000 + (\$50{,}000 \times 9.6036)$$
$$= \$530{,}180$$

$530,180, the present value in this example, is greater than $490,905, the present value in the earlier lottery example. This is to be expected because the annuity of the current example begins earlier. An annuity with an immediate initial payment is called an *annuity in advance* or, more commonly, an *annuity due*. Always remember that Equation 4.15 and Table A.2 in this book refer to an *ordinary annuity*.

Trick 3: The Infrequent Annuity The following example treats an annuity with payments occurring less frequently than once a year.

EXAMPLE 4.23

Infrequent Annuities Ann Chen receives an annuity of $450, payable once every two years. The annuity stretches out over 20 years. The first payment occurs at date 2—that is, two years from today. The annual interest rate is 6 percent.

The trick is to determine the interest rate over a two-year period. The interest rate over two years is:

$$(1.06 \times 1.06) - 1 = 12.36\%$$

That is, $100 invested over two years will yield $112.36.

What we want is the present value of a $450 annuity over 10 periods, with an interest rate of 12.36 percent per period:

$$\$450\left[\frac{1 - \dfrac{1}{(1+.1236)^{10}}}{.1236}\right] = \$450 \times A^{10}_{.1236} = \$2{,}505.57$$

Trick 4: Equating Present Value of Two Annuities The following example equates the present value of inflows with the present value of outflows.

EXAMPLE 4.24

Working with Annuities Harold and Helen Nash are saving for the college education of their newborn daughter, Susan. The Nashes estimate that college expenses will run $30,000 per year when their daughter reaches college in 18 years. The annual interest rate over the next few decades will be 14 percent. How much money must they deposit in the bank each year so that their daughter will be completely supported through four years of college?

To simplify the calculations, we assume that Susan is born today. Her parents will make the first of her four annual tuition payments on her 18th birthday. They will make equal bank deposits on each of her first 17 birthdays, but no deposit at date 0. This is illustrated as follows:

Date 0	1	2	...	17	18	19	20	21
Susan's birth	Parents' 1st deposit	Parents' 2nd deposit	...	Parents' 17th and last deposit	Tuition payment 1	Tuition payment 2	Tuition payment 3	Tuition payment 4

Mr. and Ms. Nash will be making deposits to the bank over the next 17 years. They will be withdrawing $30,000 per year over the following four years. We can be sure they will be able to withdraw fully $30,000 per year if the present value of the deposits is equal to the present value of the four $30,000 withdrawals.

This calculation requires three steps. The first two determine the present value of the withdrawals. The final step determines yearly deposits that will have a present value equal to that of the withdrawals.

1. We calculate the present value of the four years at college using the annuity formula:

$$\$30{,}000 \times \left[\frac{1 - \dfrac{1}{(1.14)^4}}{.14}\right] = \$30,000 \times A^4_{.14}$$
$$= \$30,000 \times 2.9137 = \$87{,}411$$

   We assume that Susan enters college on her 18th birthday. Given our discussion in Trick 1, $87,411 represents the present value at date 17.

2. We calculate the present value of the college education at date 0 as:

$$\frac{\$87{,}411}{(1.14)^{17}} = \$9{,}422.91$$

3. Assuming that Harold and Helen Nash make deposits to the bank at the end of each of the 17 years, we calculate the annual deposit that will yield a present value of all deposits of $9,422.91. This is calculated as:

$$C \times A^{17}_{.14} = \$9{,}422.91$$

   Because $A^{17}_{.14} = 6.3729$,

$$C = \frac{\$9{,}422.91}{6.3729} = \$1{,}478.59$$

Thus deposits of $1,478.59 made at the end of each of the first 17 years and invested at 14 percent will provide enough money to make tuition payments of $30,000 over the following four years.

An alternative method in Example 4.24 would be to (1) calculate the present value of the tuition payments at Susan's 18th birthday and (2) calculate annual deposits so that the future value of the deposits at her 18th birthday equals the present value of the tuition payments at that date. Although this technique can also provide the right answer, we have found that it is more likely to lead to errors. Therefore, we equate only present values in our presentation.

Growing Annuity

Cash flows in business are likely to grow over time, due either to real growth or to inflation. The growing perpetuity, which assumes an infinite number of cash flows, provides one formula to handle this growth. We now consider a **growing annuity**, which is a *finite* number of growing cash flows. Because perpetuities of any kind are rare, a formula for a growing annuity would be useful indeed. Here is the formula:

Formula for Present Value of Growing Annuity:

$$\mathrm{PV} = C\left[\frac{1}{r-g} - \frac{1}{r-g}\times\left(\frac{1+g}{1+r}\right)^{T}\right] = C\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{T}}{r-g}\right] \tag{4.17}$$

As before, C is the payment to occur at the end of the first period, r is the interest rate, g is the rate of growth per period, expressed as a percentage, and T is the number of periods for the annuity.

EXAMPLE 4.25

Growing Annuities Stuart Gabriel, a second-year MBA student, has just been offered a job at $80,000 a year. He anticipates his salary increasing by 9 percent a year until his retirement in 40 years. Given an interest rate of 20 percent, what is the present value of his lifetime salary?

We simplify by assuming he will be paid his $80,000 salary exactly one year from now, and that his salary will continue to be paid in annual installments. The appropriate discount rate is 20 percent. From Equation 4.17, the calculation is:

$$\text{Present value of Stuart's lifetime salary} = \$80{,}000 \times \left[\frac{1-\left(\frac{1.09}{1.20}\right)^{40}}{.20-.09}\right] = \$711{,}730.71$$

Though the growing annuity is quite useful, it is more tedious than the other simplifying formulas. Whereas most sophisticated calculators have special programs for perpetuity, growing perpetuity, and annuity, there is no special program for a growing annuity. Hence, we must calculate all the terms in Equation 4.17 directly.

EXAMPLE 4.26

More Growing Annuities In our previous example, Helen and Harold Nash planned to make 17 identical payments to fund the college education of their daughter, Susan. Alternatively, imagine that they planned to increase their payments at 4 percent per year. What would their first payment be?

The first two steps of the previous Nash family example showed that the present value of the college costs was $9,422.91. These two steps would be the same here. However, the third step must be altered. Now we must ask, How much should their first payment be so that, if payments increase by 4 percent per year, the present value of all payments will be $9,422.91?

We set the growing annuity formula equal to $9,422.91 and solve for C:

$$C\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{T}}{r-g}\right] = C\left[\frac{1-\left(\frac{1.04}{1.14}\right)^{17}}{.14-.04}\right] = \$9{,}422.91$$

Here, C = $1,192.78. Thus, the deposit on their daughter's first birthday is $1,192.78, the deposit on the second birthday is $1,240.49 (= 1.04 × $1,192.78), and so on.

4.5 What Is a Firm Worth?

Suppose you are a business appraiser trying to determine the value of small companies. How can you determine what a firm is worth? One way to think about the question of how much a firm is worth is to calculate the present value of its future cash flows.

Let us consider the example of a firm that is expected to generate net cash flows (cash inflows minus cash outflows) of $5,000 in the first year and $2,000 for each of the next five years. The firm can be sold for $10,000 seven years from now. The owners of the firm would like to be able to make 10 percent on their investment in the firm.

The value of the firm is found by multiplying the net cash flows by the appropriate present value factor. The value of the firm is simply the sum of the present values of the individual net cash flows.

The present value of the net cash flows is given next.

The Present Value of the Firm

End of Year	Net Cash Flow of the Firm	Present Value Factor (10%)	Present Value of Net Cash Flows
1	$ 5,000	.90909	$ 4,545.45
2	2,000	.82645	1,652.90
3	2,000	.75131	1,502.62
4	2,000	.68301	1,366.02
5	2,000	.62092	1,241.84
6	2,000	.56447	1,128.94
7	10,000	.51316	5,131.58
		Present value of firm	$16,569.35

We can also use the simplifying formula for an annuity:

$$\frac{\$5{,}000}{1.1} + \frac{(2{,}000 \times A_{.10}^{5})}{1.1} + \frac{10{,}000}{(1.1)^7} = \$16{,}569.35$$

Suppose you have the opportunity to acquire the firm for $12,000. Should you acquire the firm? The answer is yes because the NPV is positive:

$$\text{NPV} = \text{PV} - \text{Cost}$$
$$\$4{,}569.35 = \$16{,}569.35 - \$12{,}000$$

The incremental value (NPV) of acquiring the firm is $4,569.35.

EXAMPLE 4.27

Firm Valuation The Trojan Pizza Company is contemplating investing $1 million in four new outlets in Los Angeles. Andrew Lo, the firm's chief financial officer (CFO), has estimated that the investments will pay out cash flows of $200,000 per year for nine years and nothing thereafter. (The cash flows will occur at the end of each year and there will be no cash flow after year 9.) Mr. Lo has determined that the relevant discount rate for this investment is 15 percent. This is the rate of return that the firm can earn at comparable projects. Should the Trojan Pizza Company make the investments in the new outlets?

(continued)

The decision can be evaluated as follows:

$$\begin{aligned} \text{NPV} &= -\$1{,}000{,}000 + \frac{\$200{,}000}{1.15} + \frac{\$200{,}000}{(1.15)^2} + \cdots + \frac{\$200{,}000}{(1.15)^9} \\ &= -\$1{,}000{,}000 + \$200{,}000 \times A^{9}_{.15} \\ &= -\$1{,}000{,}000 + \$954{,}316.78 \\ &= -\$45{,}683.22 \end{aligned}$$

The present value of the four new outlets is only $954,316.78. The outlets are worth less than they cost. The Trojan Pizza Company should not make the investment because the NPV is −$45,683.22. If the Trojan Pizza Company requires a 15 percent rate of return, the new outlets are not a good investment.

Summary and Conclusions

1. Two basic concepts, *future value* and *present value,* were introduced in the beginning of this chapter. With a 10 percent interest rate, an investor with $1 today can generate a future value of $1.10 in a year, $1.21 [$=\$1 \times (1.10)^2$] in two years, and so on. Conversely, present value analysis places a current value on a future cash flow. With the same 10 percent interest rate, a dollar to be received in one year has a present value of $0.909 (=$1/1.10) in year 0. A dollar to be received in two years has a present value of $0.826 [$=\$1/(1.10)^2$].
2. We commonly express an interest rate as, say, 12 percent per year. However, we can speak of the interest rate as 3 percent per quarter. Although the stated annual interest rate remains 12 percent (=3 percent × 4), the effective annual interest rate is 12.55 percent [$=(1.03)^4 - 1$]. In other words, the compounding process increases the future value of an investment. The limiting case is continuous compounding, where funds are assumed to be reinvested every infinitesimal instant.
3. A basic quantitative technique for financial decision making is net present value analysis. The net present value formula for an investment that generates cash flows (C_i) in future periods is:

$$\text{NPV} = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = -C_0 + \sum_{i=1}^{T} \frac{C_i}{(1+r)^i}$$

   The formula assumes that the cash flow at date 0 is the initial investment (a cash outflow).
4. Frequently, the actual calculation of present value is long and tedious. The computation of the present value of a long-term mortgage with monthly payments is a good example of this. We presented four simplifying formulas:

$$\text{Perpetuity: PV} = \frac{C}{r}$$

$$\text{Growing perpetuity: PV} = \frac{C}{r-g}$$

$$\text{Annuity: PV} = C\left[\frac{1 - \dfrac{1}{(1+r)^T}}{r}\right]$$

$$\text{Growing annuity: PV} = C\left[\frac{1 - \left(\dfrac{1+g}{1+r}\right)^T}{r-g}\right]$$

5. We stressed a few practical considerations in the application of these formulas:
   - **a.** The numerator in each of the formulas, C, is the cash flow to be received *one full period hence.*
   - **b.** Cash flows are generally irregular in practice. To avoid unwieldy problems, assumptions to create more regular cash flows are made both in this textbook and in the real world.
   - **c.** A number of present value problems involve annuities (or perpetuities) beginning a few periods hence. Students should practice combining the annuity (or perpetuity) formula with the discounting formula to solve these problems.
   - **d.** Annuities and perpetuities may have periods of every two or every n years, rather than once a year. The annuity and perpetuity formulas can easily handle such circumstances.
   - **e.** We frequently encounter problems where the present value of one annuity must be equated with the present value of another annuity.

Concept Questions

1. **Compounding and Period** As you increase the length of time involved, what happens to future values? What happens to present values?
2. **Interest Rates** What happens to the future value of an annuity if you increase the rate r? What happens to the present value?
3. **Present Value** Suppose two athletes sign 10-year contracts for $80 million. In one case, we're told that the $80 million will be paid in 10 equal installments. In the other case, we're told that the $80 million will be paid in 10 installments, but the installments will increase by 5 percent per year. Who got the better deal?
4. **APR and EAR** Should lending laws be changed to require lenders to report EARs instead of APRs? Why or why not?
5. **Time Value** On subsidized Stafford loans, a common source of financial aid for college students, interest does not begin to accrue until repayment begins. Who receives a bigger subsidy, a freshman or a senior? Explain.

Use the following information for the next five questions:

On December 2, 1982, General Motors Acceptance Corporation (GMAC), a subsidiary of General Motors, offered some securities for sale to the public. Under the terms of the deal, GMAC promised to repay the owner of one of these securities $10,000 on December 1, 2012, but the investors would receive nothing until then. Investors paid GMAC $500 for each of these securities on December 2, 1982, for the promise of a $10,000 payment 30 years later.

6. **Time Value of Money** Why would GMAC be willing to accept such a small amount today ($500) in exchange for a promise to repay 20 times that amount ($10,000) in the future?
7. **Call Provisions** GMAC has the right to buy back the securities anytime it wishes by paying $10,000 (this is a term of this particular deal). What impact does this feature have on the desirability of this security as an investment?
8. **Time Value of Money** Would you be willing to pay $500 today in exchange for $10,000 in 30 years? What would be the key considerations in answering yes or no? Would your answer depend on who is making the promise to repay?
9. **Investment Comparison** Suppose that when GMAC offered the security for $500, the U.S. Treasury had offered an essentially identical security. Do you think it would have had a higher or lower price? Why?
10. **Length of Investment** The GMAC security is bought and sold on the New York Stock Exchange. If you looked at the price today, do you think the price would exceed the $500 original price? Why? If you looked in the year 2010, do you think the price would be higher or lower than today's price? Why?

Questions and Problems

BASIC
(Questions 1–20)

1. **Simple Interest versus Compound Interest** First City Bank pays 7 percent simple interest on its savings account balances, whereas Second City Bank pays 7 percent interest compounded annually. If you made a $5,000 deposit in each bank, how much more money would you earn from your Second City Bank account at the end of 10 years?

2. **Calculating Future Values** Compute the future value of $1,000 compounded annually for
   a. 10 years at 5 percent.
   b. 10 years at 7 percent.
   c. 20 years at 5 percent.
   d. Why is the interest earned in part (c) not twice the amount earned in part (a)?

3. **Calculating Present Values** For each of the following, compute the present value:

Present Value	Years	Interest Rate	Future Value
	6	5%	$ 15,451
	9	11	51,557
	18	16	886,073
	23	19	550,164

4. **Calculating Interest Rates** Solve for the unknown interest rate in each of the following:

Present Value	Years	Interest Rate	Future Value
$ 265	2		$ 307
360	9		896
39,000	15		162,181
46,523	30		483,500

5. **Calculating the Number of Periods** Solve for the unknown number of years in each of the following:

Present Value	Years	Interest Rate	Future Value
$ 625		8%	$ 1,284
810		7	4,341
18,400		21	402,662
21,500		29	173,439

6. **Calculating the Number of Periods** At 7 percent interest, how long does it take to double your money? To quadruple it?

7. **Calculating Present Values** Imprudential, Inc., has an unfunded pension liability of $800 million that must be paid in 20 years. To assess the value of the firm's stock, financial analysts want to discount this liability back to the present. If the relevant discount rate is 9.5 percent, what is the present value of this liability?

8. **Calculating Rates of Return** Although appealing to more refined tastes, art as a collectible has not always performed so profitably. During 2003, Sotheby's sold the Edgar Degas bronze sculpture *Petite Danseuse de Quartorze Ans* at auction for a price of $10,311,500. Unfortunately for the previous owner, he had purchased it in 1999 at a price of $12,377,500. What was his annual rate of return on this sculpture?

9. **Perpetuities** An investor purchasing a British consol is entitled to receive annual payments from the British government forever. What is the price of a consol that pays $120 annually if the next payment occurs one year from today? The market interest rate is 15 percent.

10. **Continuous Compounding** Compute the future value of $1,000 continuously compounded for
    a. 5 years at a stated annual interest rate of 12 percent.
    b. 3 years at a stated annual interest rate of 10 percent.
    c. 10 years at a stated annual interest rate of 5 percent.
    d. 8 years at a stated annual interest rate of 7 percent.

11. **Present Value and Multiple Cash Flows** Conoly Co. has identified an investment project with the following cash flows. If the discount rate is 10 percent, what is the present value of these cash flows? What is the present value at 18 percent? At 24 percent?

Year	Cash Flow
1	$1,200
2	600
3	855
4	1,480

12. **Present Value and Multiple Cash Flows** Investment X offers to pay you $4,000 per year for nine years, whereas Investment Y offers to pay you $6,000 per year for five years. Which of these cash flow streams has the higher present value if the discount rate is 5 percent? If the discount rate is 22 percent?

13. **Calculating Annuity Present Value** An investment offers $3,600 per year for 15 years, with the first payment occurring one year from now. If the required return is 10 percent, what is the value of the investment? What would the value be if the payments occurred for 40 years? For 75 years? Forever?

14. **Calculating Perpetuity Values** The Perpetual Life Insurance Co. is trying to sell you an investment policy that will pay you and your heirs $15,000 per year forever. If the required return on this investment is 8 percent, how much will you pay for the policy? Suppose the Perpetual Life Insurance Co. told you the policy costs $195,000. At what interest rate would this be a fair deal?

15. **Calculating EAR** Find the EAR in each of the following cases:

Stated Rate (APR)	Number of Times Compounded	Effective Rate (EAR)
11%	Quarterly	
7	Monthly	
9	Daily	
17	Infinite	

16. **Calculating APR** Find the APR, or stated rate, in each of the following cases:

Stated Rate (APR)	Number of Times Compounded	Effective Rate (EAR)
	Semiannually	8.1%
	Monthly	7.6
	Weekly	16.8
	Infinite	26.2

17. **Calculating EAR** First National Bank charges 12.2 percent compounded monthly on its business loans. First United Bank charges 12.4 percent compounded semiannually. As a potential borrower, to which bank would you go for a new loan?

18. **Interest Rates** Well-known financial writer Andrew Tobias argues that he can earn 177 percent per year buying wine by the case. Specifically, he assumes that he will consume one $10 bottle of fine Bordeaux per week for the next 12 weeks. He can either pay $10 per week or buy a case of 12 bottles today. If he buys the case, he receives a 10 percent discount and, by doing so, earns the 177 percent. Assume he buys the wine and consumes the first bottle today. Do you agree with his analysis? Do you see a problem with his numbers?

19. **Calculating Number of Periods** One of your customers is delinquent on his accounts payable balance. You've mutually agreed to a repayment schedule of $500 per month. You will charge .9 percent per month interest on the overdue balance. If the current balance is $16,500, how long will it take for the account to be paid off?

20. **Calculating EAR** Friendly's Quick Loans, Inc., offers you "three for four or I knock on your door." This means you get $3 today and repay $4 when you get your paycheck in one week (or else). What's the effective annual return Friendly's earns on this lending business? If you were brave enough to ask, what APR would Friendly's say you were paying?

INTERMEDIATE (Questions 21–50)

21. **Future Value** What is the future value in three years of $1,000 invested in an account with a stated annual interest rate of 8 percent,
    - **a.** Compounded annually?
    - **b.** Compounded semiannually?
    - **c.** Compounded monthly?
    - **d.** Compounded continuously?
    - **e.** Why does the future value increase as the compounding period shortens?

22. **Simple Interest versus Compound Interest** First Simple Bank pays 8 percent simple interest on its investment accounts. If First Complex Bank pays interest on its accounts compounded annually, what rate should the bank set if it wants to match First Simple Bank over an investment horizon of 10 years?

23. **Calculating Annuities** You are planning to save for retirement over the next 30 years. To do this, you will invest $700 a month in a stock account and $300 a month in a bond account. The return of the stock account is expected to be 11 percent, and the bond account will pay 7 percent. When you retire, you will combine your money into an account with a 9 percent return. How much can you withdraw each month from your account assuming a 25-year withdrawal period?

24. **Calculating Rates of Return** Suppose an investment offers to triple your money in 12 months (don't believe it). What rate of return per quarter are you being offered?

25. **Calculating Rates of Return** You're trying to choose between two different investments, both of which have up-front costs of $50,000. Investment G returns $85,000 in five years. Investment H returns $175,000 in 11 years. Which of these investments has the higher return?

26. **Growing Perpetuities** Mark Weinstein has been working on an advanced technology in laser eye surgery. His technology will be available in the near term. He anticipates his first annual cash flow from the technology to be $200,000, received two years from today. Subsequent annual cash flows will grow at 5 percent in perpetuity. What is the present value of the technology if the discount rate is 10 percent?

27. **Perpetuities** A prestigious investment bank designed a new security that pays a quarterly dividend of $10 in perpetuity. The first dividend occurs one quarter from today. What is the price of the security if the stated annual interest rate is 12 percent, compounded quarterly?

28. **Annuity Present Values** What is the present value of an annuity of $2,000 per year, with the first cash flow received three years from today and the last one received 22 years from today? Use a discount rate of 8 percent.

29. **Annuity Present Values** What is the value today of a 15-year annuity that pays $500 a year? The annuity's first payment occurs at the end of year 6. The annual interest rate is 12 percent for years 1 through 5, and 15 percent thereafter.

30. **Balloon Payments** Mike Bayles has just arranged to purchase a $400,000 vacation home in the Bahamas with a 20 percent down payment. The mortgage has an 8 percent stated

annual interest rate, compounded monthly, and calls for equal monthly payments over the next 30 years. His first payment will be due one month from now. However, the mortgage has an eight-year balloon payment, meaning that the balance of the loan must be paid off at the end of year 8. There were no other transaction costs or finance charges. How much will Mike's balloon payment be in eight years?

31. **Calculating Interest Expense** You receive a credit card application from Shady Banks Savings and Loan offering an introductory rate of 1.90 percent per year, compounded monthly for the first six months, increasing thereafter to 16 percent compounded monthly. Assuming you transfer the $4,000 balance from your existing credit card and make no subsequent payments, how much interest will you owe at the end of the first year?

32. **Perpetuities** Barrett Pharmaceuticals is considering a drug project that costs $240,000 today and is expected to generate end-of-year annual cash flows of $21,000, forever. At what discount rate would Barrett be indifferent between accepting or rejecting the project?

33. **Growing Annuity** Southern California Publishing Company is trying to decide whether to revise its popular textbook, *Financial Psychoanalysis Made Simple.* The company has estimated that the revision will cost $50,000. Cash flows from increased sales will be $12,000 the first year. These cash flows will increase by 6 percent per year. The book will go out of print five years from now. Assume that the initial cost is paid now and revenues are received at the end of each year. If the company requires an 11 percent return for such an investment, should it undertake the revision?

34. **Growing Annuity** Your job pays you only once a year for all the work you did over the previous 12 months. Today, December 31, you just received your salary of $50,000, and you plan to spend all of it. However, you want to start saving for retirement beginning next year. You have decided that one year from today you will begin depositing 2 percent of your annual salary in an account that will earn 8 percent per year. Your salary will increase at 4 percent per year throughout your career. How much money will you have on the date of your retirement 40 years from today?

35. **Present Value and Interest Rates** What is the relationship between the value of an annuity and the level of interest rates? Suppose you just bought a 10-year annuity of $5,000 per year at the current interest rate of 10 percent per year. What happens to the value of your investment if interest rates suddenly drop to 5 percent? What if interest rates suddenly rise to 15 percent?

36. **Calculating the Number of Payments** You're prepared to make monthly payments of $125, beginning at the end of this month, into an account that pays 10 percent interest compounded monthly. How many payments will you have made when your account balance reaches $20,000?

37. **Calculating Annuity Present Values** You want to borrow $45,000 from your local bank to buy a new sailboat. You can afford to make monthly payments of $950, but no more. Assuming monthly compounding, what is the highest APR you can afford on a 60-month loan?

38. **Calculating Loan Payments** You need a 30-year, fixed-rate mortgage to buy a new home for $200,000. Your mortgage bank will lend you the money at a 6.8 percent APR for this 360-month loan. However, you can only afford monthly payments of $1,000, so you offer to pay off any remaining loan balance at the end of the loan in the form of a single balloon payment. How large will this balloon payment have to be for you to keep your monthly payments at $1,000?

39. **Present and Future Values** The present value of the following cash flow stream is $5,979 when discounted at 10 percent annually. What is the value of the missing cash flow?

Year	Cash Flow
1	$1,000
2	?
3	2,000
4	2,000

40. **Calculating Present Values** You just won the TVM Lottery. You will receive $1 million today plus another 10 annual payments that increase by $400,000 per year. Thus, in one year you receive $1.4 million. In two years, you get $1.8 million, and so on. If the appropriate interest rate is 10 percent, what is the present value of your winnings?

41. **EAR versus APR** You have just purchased a new warehouse. To finance the purchase, you've arranged for a 30-year mortgage for 80 percent of the $1,600,000 purchase price. The monthly payment on this loan will be $10,000. What is the APR on this loan? The EAR?

42. **Present Value and Break-Even Interest** Consider a firm with a contract to sell an asset for $115,000 three years from now. The asset costs $72,000 to produce today. Given a relevant discount rate on this asset of 13 percent per year, will the firm make a profit on this asset? At what rate does the firm just break even?

43. **Present Value and Multiple Cash Flows** What is the present value of $2,000 per year, at a discount rate of 12 percent, if the first payment is received 9 years from now and the last payment is received 25 years from now?

44. **Variable Interest Rates** A 15-year annuity pays $1,500 per month, and payments are made at the end of each month. If the interest rate is 15 percent compounded monthly for the first seven years, and 12 percent compounded monthly thereafter, what is the present value of the annuity?

45. **Comparing Cash Flow Streams** You have your choice of two investment accounts. Investment A is a 15-year annuity that features end-of-month $1,000 payments and has an interest rate of 10.5 percent compounded monthly. Investment B is a 9 percent continuously compounded lump-sum investment, also good for 15 years. How much money would you need to invest in B today for it to be worth as much as Investment A 15 years from now?

46. **Calculating Present Value of a Perpetuity** Given an interest rate of 6.5 percent per year, what is the value at date $t = 7$ of a perpetual stream of $3,000 payments that begins at date $t = 15$?

47. **Calculating EAR** A local finance company quotes a 14 percent interest rate on one-year loans. So, if you borrow $20,000, the interest for the year will be $2,800. Because you must repay a total of $22,800 in one year, the finance company requires you to pay $22,800/12, or $1,900, per month over the next 12 months. Is this a 14 percent loan? What rate would legally have to be quoted? What is the effective annual rate?

48. **Calculating Present Values** A 5-year annuity of ten $6,000 semiannual payments will begin 9 years from now, with the first payment coming 9.5 years from now. If the discount rate is 12 percent compounded monthly, what is the value of this annuity five years from now? What is the value three years from now? What is the current value of the annuity?

49. **Calculating Annuities Due** As discussed in the text, an ordinary annuity assumes equal payments at the end of each period over the life of the annuity. An *annuity due* is the same thing except the payments occur at the beginning of each period instead. Thus, a three-year annual annuity due would have periodic payment cash flows occurring at Years 0, 1, and 2, whereas a three-year annual ordinary annuity would have periodic payment cash flows occurring at Years 1, 2, and 3.

a. At a 9.5 percent annual discount rate, find the present value of a six-year ordinary annuity contract of $525 payments.

b. Find the present value of the same contract if it is an annuity due.

50. **Calculating Annuities Due** You want to buy a new sports car from Muscle Motors for $56,000. The contract is in the form of a 48-month annuity due at an 8.15 percent APR. What will your monthly payment be?

CHALLENGE (Questions 51–76)

51. **Calculating Annuities Due** You want to lease a set of golf clubs from Pings Ltd. The lease contract is in the form of 24 equal monthly payments at a 12 percent stated annual interest rate, compounded monthly. Because the clubs cost $4,000 retail, Pings wants the PV of the lease payments to equal $4,000. Suppose that your first payment is due immediately. What will your monthly lease payments be?

52. **Annuities** You are saving for the college education of your two children. They are two years apart in age; one will begin college 15 years from today and the other will begin 17 years from today. You estimate your children's college expenses to be $23,000 per year per child, payable at the beginning of each school year. The annual interest rate is 6.5 percent. How much money must you deposit in an account each year to fund your children's education? Your deposits begin one year from today. You will make your last deposit when your oldest child enters college. Assume four years of college.

53. **Growing Annuities** Tom Adams has received a job offer from a large investment bank as a clerk to an associate banker. His base salary will be $35,000. He will receive his first annual salary payment one year from the day he begins to work. In addition, he will get an immediate $10,000 bonus for joining the company. His salary will grow at 4 percent each year. Each year he will receive a bonus equal to 10 percent of his salary. Mr. Adams is expected to work for 25 years. What is the present value of the offer if the discount rate is 12 percent?

54. **Calculating Annuities** You have recently won the super jackpot in the Washington State Lottery. On reading the fine print, you discover that you have the following two options:
    a. You will receive 31 annual payments of $160,000, with the first payment being delivered today. The income will be taxed at a rate of 28 percent. Taxes will be withheld when the checks are issued.
    b. You will receive $446,000 now, and you will not have to pay taxes on this amount. In addition, beginning one year from today, you will receive $101,055 each year for 30 years. The cash flows from this annuity will be taxed at 28 percent.

    Using a discount rate of 10 percent, which option should you select?

55. **Calculating Growing Annuities** You have 30 years left until retirement and want to retire with $1 million. Your salary is paid annually, and you will receive $55,000 at the end of the current year. Your salary will increase at 3 percent per year, and you can earn a 10 percent return on the money you invest. If you save a constant percentage of your salary, what percentage of your salary must you save each year?

56. **Balloon Payments** On September 1, 2004, Susan Chao bought a motorcycle for $15,000. She paid $1,000 down and financed the balance with a five-year loan at a stated annual interest rate of 9.6 percent, compounded monthly. She started the monthly payments exactly one month after the purchase (i.e., October 1, 2004). Two years later, at the end of October 2006, Susan got a new job and decided to pay off the loan. If the bank charges her a 1 percent prepayment penalty based on the loan balance, how much must she pay the bank on November 1, 2006?

57. **Calculating Annuity Values** Bilbo Baggins wants to save money to meet three objectives. First, he would like to be able to retire 30 years from now with a retirement income of $25,000 per month for 20 years, with the first payment received 30 years and 1 month from now. Second, he would like to purchase a cabin in Rivendell in 10 years at an estimated cost of $350,000. Third, after he passes on at the end of the 20 years of withdrawals, he would like to leave an inheritance of $750,000 to his nephew Frodo. He can afford to save $2,100 per month for the next 10 years. If he can earn an 11 percent EAR before he retires and an 8 percent EAR after he retires, how much will he have to save each month in years 11 through 30?

58. **Calculating Annuity Values** After deciding to buy a new car, you can either lease the car or purchase it with a three-year loan. The car you wish to buy costs $35,000. The dealer has a special leasing arrangement where you pay $1 today and $450 per month for the next three years. If you purchase the car, you will pay it off in monthly payments over the next three years at an 8 percent APR. You believe that you will be able to sell the car for $23,000 in three years. Should you buy or lease the car? What break-even resale price in three years would make you indifferent between buying and leasing?

59. **Calculating Annuity Values** An All-Pro defensive lineman is in contract negotiations. The team has offered the following salary structure:

Time	Salary
0	$8,000,000
1	$4,000,000
2	$4,800,000
3	$5,700,000
4	$6,400,000
5	$7,000,000
6	$7,500,000

All salaries are to be paid in a lump sum. The player has asked you as his agent to renegotiate the terms. He wants a $9 million signing bonus payable today and a contract value increase of $750,000. He also wants an equal salary paid every three months, with the first paycheck three months from now. If the interest rate is 4.5 percent compounded daily, what is the amount of his quarterly check? Assume 365 days in a year.

60. **Discount Interest Loans** This question illustrates what is known as *discount interest.* Imagine you are discussing a loan with a somewhat unscrupulous lender. You want to borrow $20,000 for one year. The interest rate is 12 percent. You and the lender agree that the interest on the loan will be $.12 \times \$20{,}000 = \$2{,}400$. So, the lender deducts this interest amount from the loan up front and gives you $17,600. In this case, we say that the discount is $2,400. What's wrong here?

61. **Calculating Annuity Values** You are serving on a jury. A plaintiff is suing the city for injuries sustained after a freak street sweeper accident. In the trial, doctors testified that it will be five years before the plaintiff is able to return to work. The jury has already decided in favor of the plaintiff. You are the foreperson of the jury and propose that the jury give the plaintiff an award to cover the following: (1) The present value of two years' back pay. The plaintiff's annual salary for the last two years would have been $40,000 and $43,000, respectively. (2) The present value of five years' future salary. You assume the salary will be $45,000 per year. (3) $100,000 for pain and suffering. (4) $20,000 for court costs. Assume that the salary payments are equal amounts paid at the end of each month. If the interest rate you choose is a 9 percent EAR, what is the size of the settlement? If you were the plaintiff, would you like to see a higher or lower interest rate?

62. **Calculating EAR with Points** You are looking at a one-year loan of $10,000. The interest rate is quoted as 10 percent plus three points. A *point* on a loan is simply 1 percent (one percentage point) of the loan amount. Quotes similar to this one are very common with home mortgages. The interest rate quotation in this example requires the borrower to pay three points to the lender up front and repay the loan later with 10 percent interest. What rate would you actually be paying here? What is the EAR for a one-year loan with a quoted interest rate of 13 percent plus two points? Is your answer affected by the loan amount?

63. **EAR versus APR** Two banks in the area offer 30-year, $200,000 mortgages at 7.5 percent and charge a $1,500 loan application fee. However, the application fee charged by Insecurity Bank and Trust is refundable if the loan application is denied, whereas that charged by I. M. Greedy and Sons Mortgage Bank is not. The current disclosure law requires that any fees that will be refunded if the applicant is rejected be included in calculating the APR, but this is not required with nonrefundable fees (presumably because refundable fees are part of the loan rather than a fee). What are the EARs on these two loans? What are the APRs?

64. **Calculating EAR with Add-On Interest** This problem illustrates a deceptive way of quoting interest rates called *add-on interest.* Imagine that you see an advertisement for Crazy Judy's Stereo City that reads something like this: "$1,000 Instant Credit! 15% Simple Interest! Three Years to Pay! Low, Low Monthly Payments!" You're not exactly sure what all this means and somebody has spilled ink over the APR on the loan contract, so you ask the manager for clarification.

Judy explains that if you borrow $1,000 for three years at 15 percent interest, in three years you will owe:

$$\$1{,}000 \times 1.15^3 = \$1{,}000 \times 1.52088 = \$1{,}520.88$$

Judy recognizes that coming up with $1,520.88 all at once might be a strain, so she lets you make "low, low monthly payments" of $1,520.88/36 = $42.25 per month, even though this is extra bookkeeping work for her.

Is this a 15 percent loan? Why or why not? What is the APR on this loan? What is the EAR? Why do you think this is called add-on interest?

65. Calculating Annuity Payments This is a classic retirement problem. A time line will help in solving it. Your friend is celebrating her 35th birthday today and wants to start saving for her anticipated retirement at age 65. She wants to be able to withdraw $90,000 from her savings account on each birthday for 15 years following her retirement; the first withdrawal will be on her 66th birthday. Your friend intends to invest her money in the local credit union, which offers 8 percent interest per year. She wants to make equal annual payments on each birthday into the account established at the credit union for her retirement fund.

a. If she starts making these deposits on her 36th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals at retirement?

b. Suppose your friend has just inherited a large sum of money. Rather than making equal annual payments, she has decided to make one lump-sum payment on her 35th birthday to cover her retirement needs. What amount does she have to deposit?

c. Suppose your friend's employer will contribute $1,500 to the account every year as part of the company's profit-sharing plan. In addition, your friend expects a $25,000 distribution from a family trust fund on her 55th birthday, which she will also put into the retirement account. What amount must she deposit annually now to be able to make the desired withdrawals at retirement?

66. Calculating the Number of Periods Your Christmas ski vacation was great, but it unfortunately ran a bit over budget. All is not lost: You just received an offer in the mail to transfer your $10,000 balance from your current credit card, which charges an annual rate of 19.2 percent, to a new credit card charging a rate of 9.2 percent. How much faster could you pay the loan off by making your planned monthly payments of $200 with the new card? What if there was a 2 percent fee charged on any balances transferred?

67. Future Value and Multiple Cash Flows An insurance company is offering a new policy to its customers. Typically the policy is bought by a parent or grandparent for a child at the child's birth. The details of the policy are as follows: The purchaser (say, the parent) makes the following six payments to the insurance company:

First birthday:	$750
Second birthday:	$750
Third birthday:	$850
Fourth birthday:	$850
Fifth birthday:	$950
Sixth birthday:	$950

After the child's sixth birthday, no more payments are made. When the child reaches age 65, he or she receives $250,000. If the relevant interest rate is 11 percent for the first six years and 7 percent for all subsequent years, is the policy worth buying?

68. Annuity Present Values and Effective Rates You have just won the lottery. You will receive $1,000,000 today, and then receive 40 payments of $500,000. These payments will start one year from now and will be paid every six months. A representative from Greenleaf Investments has offered to purchase all the payments from you for $10 million. If the appropriate interest rate is a 9 percent APR compounded daily, should you take the offer? Assume there are 12 months in a year, each with 30 days.

69. Calculating Interest Rates A financial planning service offers a college savings program. The plan calls for you to make six annual payments of $8,000 each, with the first payment occurring today, your child's 12th birthday. Beginning on your child's 18th birthday, the plan will provide $20,000 per year for four years. What return is this investment offering?

70. **Break-Even Investment Returns** Your financial planner offers you two different investment plans. Plan X is a \$10,000 annual perpetuity. Plan Y is a 10-year, \$22,000 annual annuity. Both plans will make their first payment one year from today. At what discount rate would you be indifferent between these two plans?

71. **Perpetual Cash Flows** What is the value of an investment that pays \$6,700 every *other* year forever, if the first payment occurs one year from today and the discount rate is 13 percent compounded daily? What is the value today if the first payment occurs four years from today?

72. **Ordinary Annuities and Annuities Due** As discussed in the text, an annuity due is identical to an ordinary annuity except that the periodic payments occur at the beginning of each period and not at the end of the period. Show that the relationship between the value of an ordinary annuity and the value of an otherwise equivalent annuity due is:

$$\text{Annuity due value} = \text{Ordinary annuity value} \times (1 + r)$$

Show this for both present and future values.

73. **Calculating EAR** A check-cashing store is in the business of making personal loans to walk-up customers. The store makes only one-week loans at 10 percent interest per week.

a. What APR must the store report to its customers? What is the EAR that the customers are actually paying?

b. Now suppose the store makes one-week loans at 10 percent discount interest per week (see Question 60). What's the APR now? The EAR?

c. The check-cashing store also makes one-month add-on interest loans at 9 percent discount interest per week. Thus, if you borrow \$100 for one month (four weeks), the interest will be $(\$100 \times 1.09^4) - 100 = \$41.16$. Because this is discount interest, your net loan proceeds today will be \$58.84. You must then repay the store \$100 at the end of the month. To help you out, though, the store lets you pay off this \$100 in installments of \$25 per week. What is the APR of this loan? What is the EAR?

74. **Present Value of a Growing Perpetuity** What is the equation for the present value of a growing perpetuity with a payment of C one period from today if the payments grow by C each period?

75. **Rule of 72** A useful rule of thumb for the time it takes an investment to double with discrete compounding is the "Rule of 72." To use the rule of 72, you simply divide 72 by the interest rate to determine the number of periods it takes for a value today to double. For example, if the interest rate is 6 percent, the rule of 72 say it will take $72/6 = 12$ years to double. This is approximately equal to the actual answer of 11.90 years. The Rule of 72 can also be applied to determine what interest rate is needed to double money in a specified period. This is a useful approximation for many interest rates and periods. At what rate is the Rule of 72 exact?

76. **Rule of 69.3** A corollary to the Rule of 72 is the Rule of 69.3. The Rule of 69.3 is exactly correct except for rounding when interest rates are compounded continuously. Prove the Rule of 69.3 for continuously compounded interest.

S&P Problems

STANDARD & POOR'S

www.mhhe.com/edumarketinsight

1. Under the "Excel Analytics" link find the "Mthly. Adj. Price" for Elizabeth Arden (RDEN) stock. What was your annual return over the last four years assuming you purchased the stock at the close price four years ago? (Assume no dividends were paid.) Using this same return, what price will Elizabeth Arden stock sell for five years from now? Ten years from now? What if the stock price increases at 11 percent per year?

2. **Calculating the Number of Periods** Find the monthly adjusted stock prices for Southwest Airlines (LUV). You find an analyst who projects the stock price will increase 12 percent per year for the foreseeable future. Based on the most recent monthly stock price, if the projection holds true, when will the stock price reach \$150? When will it reach \$200?

Appendix 4A Net Present Value: First Principles of Finance

To access the appendix for this chapter, please go to www.mhhe.com/rwj.

Mini Case

The MBA Decision

Ben Bates graduated from college six years ago with a finance undergraduate degree. Although he is satisfied with his current job, his goal is to become an investment banker. He feels that an MBA degree would allow him to achieve this goal. After examining schools, he has narrowed his choice to either Wilton University or Mount Perry College. Although internships are encouraged by both schools, to get class credit for the internship, no salary can be paid. Other than internships, neither school will allow its students to work while enrolled in its MBA program.

Ben currently works at the money management firm of Dewey and Louis. His annual salary at the firm is $50,000 per year, and his salary is expected to increase at 3 percent per year until retirement. He is currently 28 years old and expects to work for 35 more years. His current job includes a fully paid health insurance plan, and his current average tax rate is 26 percent. Ben has a savings account with enough money to cover the entire cost of his MBA program.

The Ritter College of Business at Wilton University is one of the top MBA programs in the country. The MBA degree requires two years of full-time enrollment at the university. The annual tuition is $60,000, payable at the beginning of each school year. Books and other supplies are estimated to cost $2,500 per year. Ben expects that after graduation from Wilton, he will receive a job offer for about $95,000 per year, with a $15,000 signing bonus. The salary at this job will increase at 4 percent per year. Because of the higher salary, his average income tax rate will increase to 31 percent.

The Bradley School of Business at Mount Perry College began its MBA program 16 years ago. The Bradley School is smaller and less well known than the Ritter College. Bradley offers an accelerated, one-year program, with a tuition cost of $75,000 to be paid upon matriculation. Books and other supplies for the program are expected to cost $3,500. Ben thinks that he will receive an offer of $78,000 per year upon graduation, with a $10,000 signing bonus. The salary at this job will increase at 3.5 percent per year. His average tax rate at this level of income will be 29 percent.

Both schools offer a health insurance plan that will cost $3,000 per year, payable at the beginning of the year. Ben also estimates that room and board expenses will cost $20,000 per year at either school. The appropriate discount rate is 6.5 percent.

1. How does Ben's age affect his decision to get an MBA?
2. What other, perhaps nonquantifiable factors affect Ben's decision to get an MBA?
3. Assuming all salaries are paid at the end of each year, what is the best option for Ben—from a strictly financial standpoint?
4. Ben believes that the appropriate analysis is to calculate the future value of each option. How would you evaluate this statement?
5. What initial salary would Ben need to receive to make him indifferent between attending Wilton University and staying in his current position?
6. Suppose, instead of being able to pay cash for his MBA, Ben must borrow the money. The current borrowing rate is 5.4 percent. How would this affect his decision?

Appendix 4A Net Present Value: First Principles of Finance

In this appendix, we show the theoretical underpinnings of the net present value rule. We first show how individuals make intertemporal consumption choices, and then we explain the net present value (NPV) rule. The appendix should appeal to students who like a theoretical model. Those of you who can accept the NPV analysis contained in Chapter 4 can skip to Chapter 5.

4A.1 Making Consumption Choices over Time

Figure 4A.1 illustrates the situation faced by a representative individual in the financial market. This person is assumed to have an income of \$50,000 this year and an income of \$60,000 next year. The market allows him not only to consume \$50,000 worth of goods this year and \$60,000 next year, but also to borrow and lend at the equilibrium interest rate.

The line *AB* in Figure 4A.1 shows all of the consumption possibilities open to the person through borrowing or lending, and the shaded area contains all of the feasible choices. Let's look at this figure more closely to see exactly why points in the shaded area are available.

We will use the letter r to denote the interest rate—the equilibrium rate—in this market. The rate is risk-free because we assume that no default can take place. Look at point A on the vertical axis of Figure 4A.1. Point A is a height of:

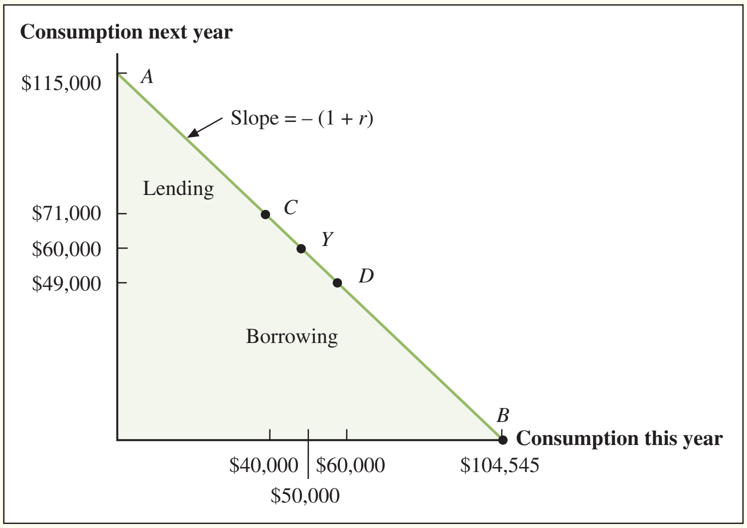
$$A = \$60{,}000 + [\$50{,}000 \times (1 + r)]$$

For example, if the rate of interest is 10 percent, then point A would be:

$$\begin{aligned} A &= \$60{,}000 + [\$50{,}000 \times (1 + 0.1)] \\ &= \$60{,}000 + \$55{,}000 \\ &= \$115{,}000 \end{aligned}$$

Point A is the maximum amount of wealth that this person can spend in the second year. He gets to point A by lending the full income that is available this year, \$50,000, and consuming none of it. In the second year, then, he will have the second year's income of \$60,000 plus the proceeds from the loan that he made in the first year, \$55,000, for a total of \$115,000.

Figure 4A.1
Intertemporal Consumption Opportunities



Now let's take a look at point *B*. Point *B* is a distance of:

$$B = \$50{,}000 + [\$60{,}000/(1 + r)]$$

along the horizontal axis. If the interest rate is 10 percent, point *B* will be:

$$\begin{aligned} B &= \$50{,}000 + [\$60{,}000/(1 + 0.1)] \\ &= \$50{,}000 + \$54{,}545 \\ &= \$104{,}545 \end{aligned}$$

(We have rounded off to the nearest dollar.)

Why do we divide next year's income of \$60,000 by $(1 + r)$, or 1.1 in the preceding computation? Point *B* represents the maximum amount available for this person to consume this year. To achieve that maximum he would borrow as much as possible and repay the loan from the income, \$60,000, that he was going to receive next year. Because \$60,000 will be available to repay the loan next year, we are asking how much he could borrow this year at an interest rate of r and still be able to repay the loan. The answer is:

$$\$60{,}000/(1 + r)$$

because if he borrows this amount, he must repay it next year with interest. Thus, next year he must repay:

$$[\$60{,}000/(1 + r)] \times (1 + r) = \$60{,}000$$

no matter what the interest rate, r, is. In our example we found that he could borrow \$54,545 and, sure enough:

$$\$54{,}545 \times 1.1 = \$60{,}000$$

(after rounding off to the nearest dollar).

Furthermore, by borrowing and lending different amounts the person can achieve any point on the line *AB*. For example, point *C* is a point where he has chosen to lend \$10,000 of today's income. This means that at point *C* he will have:

$$\text{Consumption this year at point } C = \$50{,}000 - \$10{,}000 = \$40{,}000$$

and

$$\text{Consumption next year at point } C = \$60{,}000 + [\$10{,}000 \times (1 + r)] = \$71{,}000$$

when the interest rate is 10 percent.

Similarly, at point *D* the individual has decided to borrow \$10,000 and repay the loan next year. At point *D*:

$$\text{Consumption this year at point } D = \$50{,}000 + \$10{,}000 = \$60{,}000$$

and:

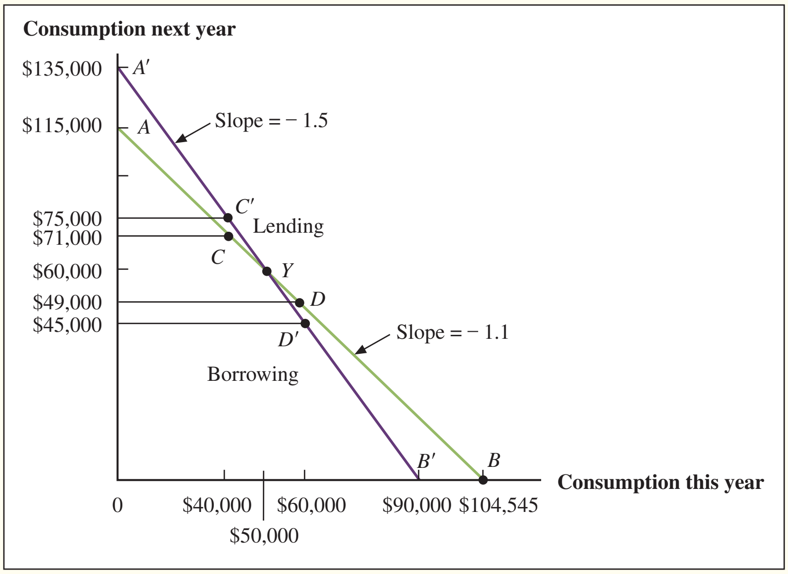
$$\text{Consumption next year at point } D = \$60{,}000 - [\$10{,}000 \times (1 + r)] = \$49{,}000$$

at an interest rate of 10 percent.

In fact, this person can consume at any point on the line *AB*. This line has a slope of $-(1 + r)$, which means that for each dollar that is added to the x coordinate along the line, $(1 + r)$ dollars are subtracted from the y coordinate. Moving along the line from point *A*, the initial point of \$50,000 this year and \$60,000 next year, toward point *B* gives the person more consumption today and less next year. In other words, moving toward point *B* is borrowing. Similarly, moving up toward point *A*, he is consuming less today and more next

Figure 4A.2

The Effect of Different Interest Rates on Consumption Opportunities



year and is lending. The line is a straight line because the individual has no effect on the interest rate. This is one of the assumptions of perfectly competitive financial markets.

Where will the person actually be? The answer to that question depends on the individual's tastes and personal situation, just as it did before there was a market. If the person is impatient, he might wish to borrow money at a point such as *D.* If he is patient, he might wish to lend some of this year's income and enjoy more consumption next year at, for example, a point such as *C.*[1]

Notice that whether we think of someone as patient or impatient depends on the interest rate he or she faces in the market. Suppose that our individual was impatient and chose to borrow $10,000 and move to point *D.* Now suppose that we raise the interest rate to 20 percent or even 50 percent. Suddenly our impatient person may become very patient and might prefer to lend some of this year's income to take advantage of the very high interest rate. The general result is depicted in Figure 4A.2. We can see that lending at point *C′* yields much greater future income and consumption possibilities than before.[2]

4A.2 Making Investment Choices

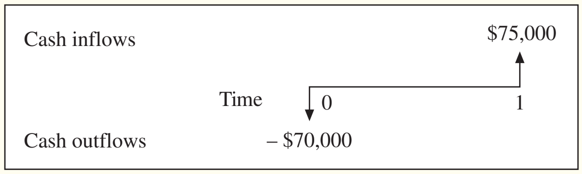
A Lending Example Consider a person who is concerned only about this year and the next. She has an income of $100,000 this year and expects to make the same amount next

[1]In this section, we are assuming a certain kind of financial market. In the language of economics, individuals who respond to rates and prices by acting as though they have no influence on them are called *price takers,* and this assumption is sometimes called the *price-taking assumption.* It is the condition of **perfectly competitive financial markets** (or, more simply, *perfect markets*). The following conditions are likely to lead to this:

1. Trading is costless. Access to the financial markets is free.
2. Information about borrowing and lending opportunities is available.
3. There are many traders, and no single trader can have a significant impact on market prices.

[2]Those familiar with consumer theory might be aware of the surprising case where raising the interest rate actually makes people borrow more or lowering the rate makes them lend more. The latter case might occur, for example, if the decline in the interest rate made the lenders have so little consumption next year that they have no choice but to lend out even more than they were lending before just to subsist. Nothing we do depends on excluding such cases, but it is much easier to ignore them, and the resulting analysis fits the real markets more closely.

Figure 4A.3
Cash Flows for Investment in Land



year. The interest rate is 10 percent. This individual is thinking about investing in a piece of land that costs $70,000. She is certain that next year the land will be worth $75,000, a sure $5,000 gain. Should she undertake the investment? This situation is described in Figure 4A.3 with the cash flow time chart.

A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $70,000 in the land, she will have $75,000 available next year. Suppose instead that she puts the same $70,000 into a loan in the financial market. At the 10 percent rate of interest this $70,000 would grow to:

$$(1 + 0.1) \times \$70{,}000 = \$77{,}000$$

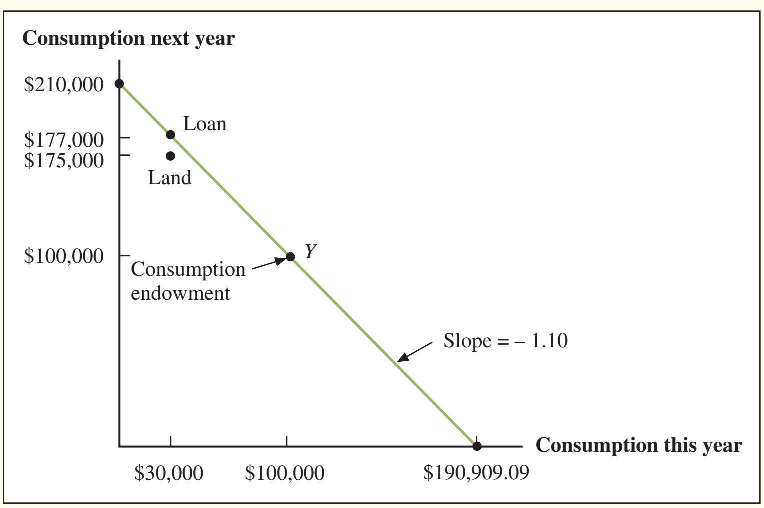
next year.

It would be foolish to buy the land when the same $70,000 investment in the financial market would beat it by $2,000 (that is, $77,000 from the loan minus $75,000 from the land investment). Figure 4A.4 illustrates this situation. Notice that the $70,000 loan gives no less income today and $2,000 more next year. This example illustrates some amazing features of the financial markets. It is remarkable to consider all of the information that we did *not* use when arriving at the decision not to invest in the land. We did not need to know how much income the person has this year or next year. We also did not need to know whether the person preferred more income this year or next.

We did not need to know any of these other facts, and more important, the person making the decision did not need to know them either. She needed only to be able to compare the investment with a relevant alternative available in the financial market. When this investment fell short of that standard—by $2,000 in the previous example—regardless of what the individual wanted to do, she knew that she should not buy the land.

A Borrowing Example Let us sweeten the deal a bit. Suppose that instead of being worth $75,000 next year, the land would be worth $80,000. What should our investor do

Figure 4A.4
Consumption Opportunities with Borrowing and Lending



now? This case is a bit more difficult. After all, even if the land seems like a good deal, this person's income this year is \$100,000. Does she really want to make a \$70,000 investment this year? Won't that leave only \$30,000 for consumption?

The answers to these questions are yes, the individual should buy the land; yes, she does want to make a \$70,000 investment this year; and, most surprising of all, even though her income is \$100,000, making the \$70,000 investment will not leave her with \$30,000 to consume this year! Now let us see how finance lets us get around the basic laws of arithmetic.

The financial markets are the key to solving our problem. First, the financial markets can be used as a standard of comparison against which any investment project must be measured. Second, they can be used as a tool to actually help the individual undertake investments. These twin features of the financial markets enable us to make the right investment decision.

Suppose that the person borrows the \$70,000 initial investment that is needed to purchase the land. Next year she must repay this loan. Because the interest rate is 10 percent, she will owe the financial market \$77,000 next year. This is depicted in Figure 4A.5. Because the land will be worth \$80,000 next year, she can sell it, pay off her debt of \$77,000, and have \$3,000 extra cash.

If she wishes, this person can now consume an extra \$3,000 worth of goods and services next year. This possibility is illustrated in Figure 4A.6. In fact, even if she wants to do all of her consuming this year, she is still better off taking the investment. All she must do is take out a loan this year and repay it from the proceeds of the land next year and profit by \$3,000.

Furthermore, instead of borrowing just the \$70,000 that she needed to purchase the land, she could have borrowed \$72,727.27. She could have used \$70,000 to buy the land and consumed the remaining \$2,727.27.

We will call \$2,727.27 the net present value of the transaction. Notice that it is equal to \$3,000 × 1/1.1. How did we figure out that this was the exact amount that she could

Figure 4A.5
Cash Flows of Borrowing to Purchase the Land

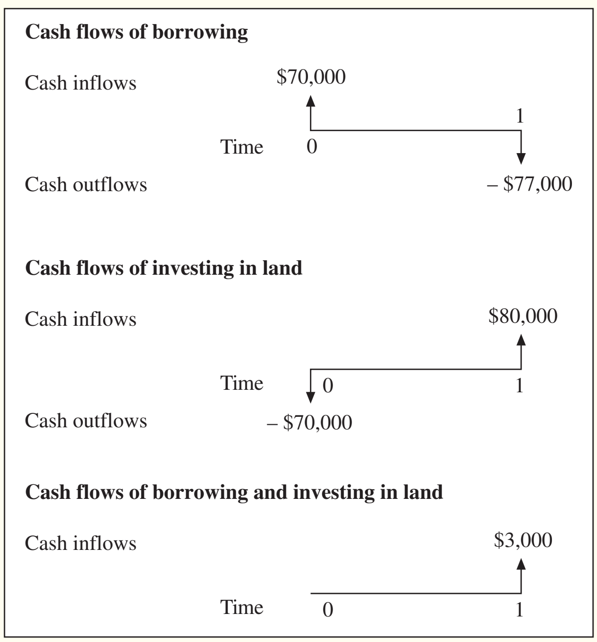
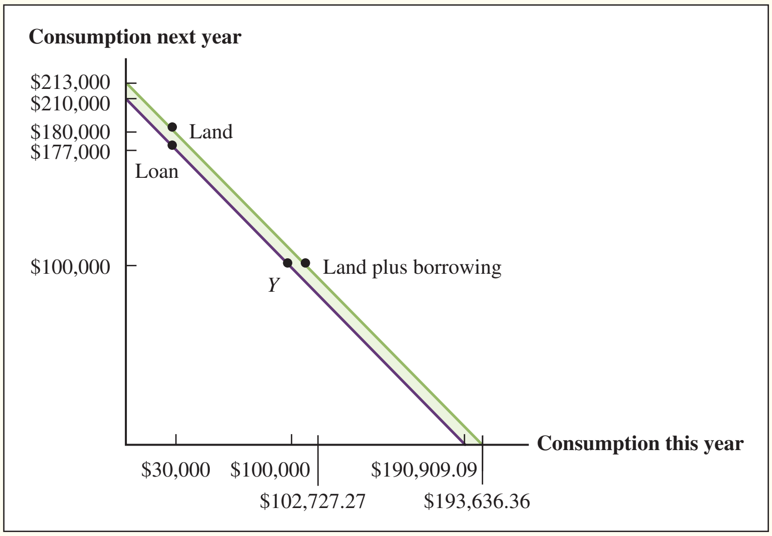


Figure 4A.6
Consumption Opportunities with Investment Opportunity and Borrowing and Lending



borrow? It was easy: If $72,727.27 is the amount that she borrows, then, because the interest rate is 10 percent, she must repay:

$$\$72{,}727.27 \times (1 + 0.1) = \$80{,}000$$

next year, and that is exactly what the land will be worth. The line through the investment position in Figure 4A.6 illustrates this borrowing possibility.

The amazing thing about both of these cases, one where the land is worth $75,000 next year and the other where it is worth $80,000 next year, is that we needed only to compare the investment with the financial markets to decide whether it was worth undertaking. This is one of the more important points in all of finance. It is true regardless of the consumption preferences of the individual. This is one of a number of **separation theorems** in finance. It states that the value of an investment to an individual is not dependent on consumption preferences. In our examples we showed that the person's decision to invest in land was not affected by consumption preferences. However, these preferences dictated whether she borrowed or lent.

4A.3 Illustrating the Investment Decision

Figure 4A.1 describes the possibilities open to a person who has an income of $50,000 this year and $60,000 next year and faces a financial market in which the interest rate is 10 percent. But, at that moment, the person has no investment possibilities beyond the 10 percent borrowing and lending that is available in the financial market.

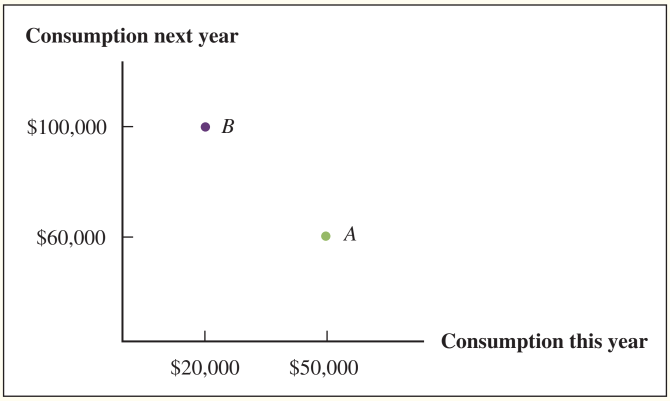
Suppose we give this person the chance to undertake an investment project that will require a $30,000 outlay of cash this year and that will return $40,000 to the investor next year. Refer to Figure 4A.1 and determine how you could include this new possibility in that figure and how you could use the figure to help you decide whether to undertake the investment.

Now look at Figure 4A.7. In Figure 4A.7 we have labeled the original point with $50,000 this year and $60,000 next year as point *A*. We have also added a new point *B*, with $20,000 available for consumption this year and $100,000 next year. The difference between point *A* and point *B* is that at point *A* the person is just where we started him off, and at point *B* the person has also decided to undertake the investment project. As a result of this decision the person at point *B* has:

$$\$50{,}000 - 30{,}000 = \$20{,}000$$

Figure 4A.7

Consumption Choices with Investment but No Financial Markets



left for consumption this year, and:

$$\$60{,}000 + 40{,}000 = \$100{,}000$$

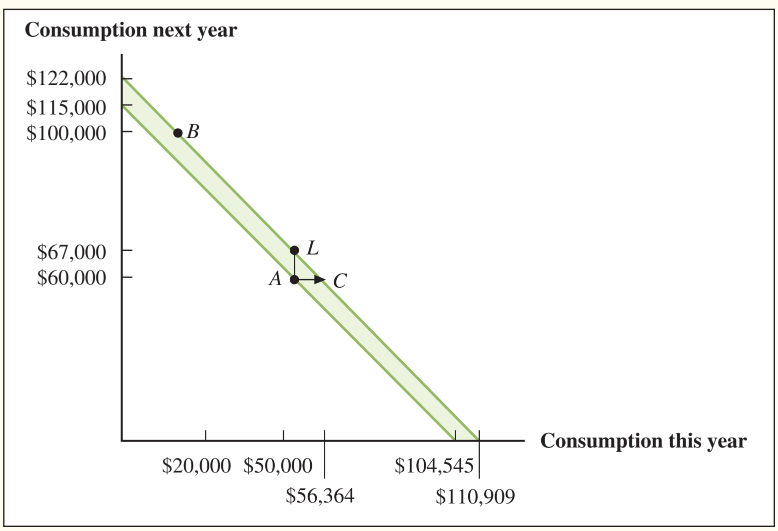
available next year. These are the coordinates of point *B*.

We must use our knowledge of the individual's borrowing and lending opportunities to decide whether to accept or reject the investment. This is illustrated in Figure 4A.8. Figure 4A.8 is similar to Figure 4A.7, but in it we have drawn a line through point *A* that shows the possibilities open to the person if he stays at point *A* and does not make the investment. This line is exactly the same as the one in Figure 4A.1. We have also drawn a parallel line through point *B* that shows the new possibilities that are available to the person if he undertakes the investment. The two lines are parallel because the slope of each is determined by the same interest rate, 10 percent. It does not matter whether the person takes the investment and goes to point *B* or does not and stays at point *A*: in the financial market, each dollar of lending is a dollar less available for consumption this year and moves him to the left by a dollar along the *x*-axis. Because the interest rate is 10 percent, the $1 loan repays $1.10 and moves him up by $1.10 along the *y*-axis.

It is easy to see from Figure 4A.8 that the investment has made the person better off. The line through point *B* is higher than the line through point *A*. Thus, no matter what pattern of consumption this person wanted this year and next, he could have more in each year if he undertook the investment.

Figure 4A.8

Consumption Choices with Investment Opportunities and Financial Markets



For example, suppose our individual wanted to consume everything this year. If he did not make the investment, the point where the line through point *A* intersected the *x*-axis would give the maximum amount of consumption he could enjoy this year. This point has $104,545 available this year. To recall how we found this figure, review the analysis of Figure 4A.1. But in Figure 4A.8 the line that goes through point *B* hits the *x*-axis at a higher point than the line that goes through point *A*. Along this line the person can have the $20,000 that is left after investing $30,000, plus all that he can borrow and repay with both next year's income and the proceeds from the investment. The total amount available to consume today is therefore:

$$\begin{aligned}&= \$50{,}000 - 30{,}000 + (\$60{,}000 + 40{,}000)/(1 + 0.1)\\ &= \$20{,}000 + (\$100{,}000/1.1)\\ &= \$110{,}909\end{aligned}$$

The additional consumption available this year from undertaking the investment and using the financial market is the difference on the *x*-axis between the points where these two lines intersect:

$$\$110{,}909 - 104{,}545 = \$6{,}364$$

This difference is an important measure of what the investment is worth to the person. It answers a variety of questions. For example, it reveals how much money we would need to give the investor this year to make him just as well off as he is with the investment.

Because the line through point *B* is parallel to the line through point *A* but has been moved over by $6,364, we know that if we were to add this amount to the investor's current income this year at point *A* and take away the investment, he would wind up on the line through point *B* and with the same possibilities. If we do this, the person will have $56,364 this year and $60,000 next year, which is the situation of the point on the line through point *B* that lies to the right of point *A* in Figure 4A.8. This is point *C*.

We could also ask a different question: How much money would we need to give the investor next year to make him just as well off as he is with the investment?

This is the same as asking how much higher the line through point *B* is than the line through point *A*. In other words, what is the difference in Figure 4A.8 between the point where the line through *A* intercepts the *y*-axis and the point where the line through *B* intercepts the *y*-axis?

The point where the line through *A* intercepts the *y*-axis shows the maximum amount the person could consume next year if all of his current income were lent out and the proceeds of the loan were consumed along with next year's income.

As we showed in our analysis of Figure 4A.1, this amount is $115,000. How does this compare with what the person can have next year if he makes the investment? By making the investment we saw that he would be at point *B*, where he has $20,000 left this year and would have $100,000 next year. By lending the $20,000 that is left this year and adding the proceeds of this loan to the $100,000, we find the line through *B* intercepts the *y*-axis at

$$(\$20{,}000 \times 1.1) + \$100{,}000 = \$122{,}000$$

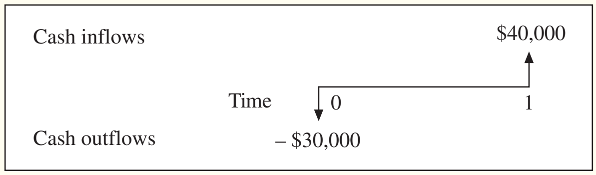
The difference between this amount and $115,000 is

$$\$122{,}000 - 115{,}000 = \$7{,}000$$

which is the answer to the question of how much we would need to give the person next year to make him as well off as he is with the investment.

Figure 4A.9

Cash Flows for the Investment Project



There is a simple relationship between these two numbers. If we multiply \$6,364 by 1.1 we get \$7,000! Consider why this must be so. The \$6,364 is the amount of extra cash we must give the person this year to substitute for having the investment. In a financial market with a 10 percent rate of interest, however, \$1 this year is worth exactly the same as \$1.10 next year. Thus, \$6,364 this year is the same as \$6,364 × 1.1 next year. In other words, the person does not care whether he has the investment, \$6,364, this year or \$6,364 × 1.1 next year. But we already showed that the investor is equally willing to have the investment and to have \$7,000 next year. This must mean that:

$$\$6{,}364 \times 1.1 = \$7{,}000$$

You can also verify this relationship between these two variables by using Figure 4A.8. Because the lines through *A* and *B* each have the same slope of −1.1, the difference of \$7,000 between where they intersect on the *x*-axis must be in the ratio of 1.1 to 1.

Now we can show you how to evaluate the investment opportunity on a stand-alone basis. Here are the relevant facts: The individual must give up \$30,000 this year to get \$40,000 next year. These cash flows are illustrated in Figure 4A.9.

The investment rule that follows from the previous analysis is the net present value (NPV) rule. Here, we convert all consumption values to the present and add them up:

$$\begin{aligned}\text{Net present value} &= -\$30{,}000 + \$40{,}000 \times (1/1.1)\\ &= -\$30{,}000 + \$36{,}364\\ &= \$6{,}364\end{aligned}$$

The future amount, \$40,000, is called the *future value* (FV).

The net present value of an investment is a simple criterion for deciding whether to undertake an investment. NPV answers the question of how much cash an investor would need to have today as a substitute for making the investment. If the net present value is positive, the investment is worth taking on because doing so is essentially the same as receiving a cash payment equal to the net present value. If the net present value is negative, taking on the investment today is equivalent to giving up some cash today, and the investment should be rejected.

We use the term *net present value* to emphasize that we are already including the current cost of the investment in determining its value and not simply what it will return. For example, if the interest rate is 10 percent and an investment of \$30,000 today will produce a total cash return of \$40,000 in one year's time, the *present value* of the \$40,000 by itself is:

$$\$40{,}000/1.1 = \$36{,}364$$

but, the *net present value* of the investment is \$36,364 minus the original investment:

$$\text{Net present value} = \$36{,}364 - \$30{,}000 = \$6{,}364$$

The present value of a future cash flow is the value of that cash flow after considering the appropriate market interest rate. The net present value of an investment is the present value of the investment's future cash flows, minus the initial cost of the investment. We have just decided that our investment is a good opportunity. It has a positive net present value because it is worth more than it costs.

In general, our results can be stated in terms of the **net present value rule**:

> An investment is worth making if it has a positive NPV. If an investment's NPV is negative, it should be rejected.

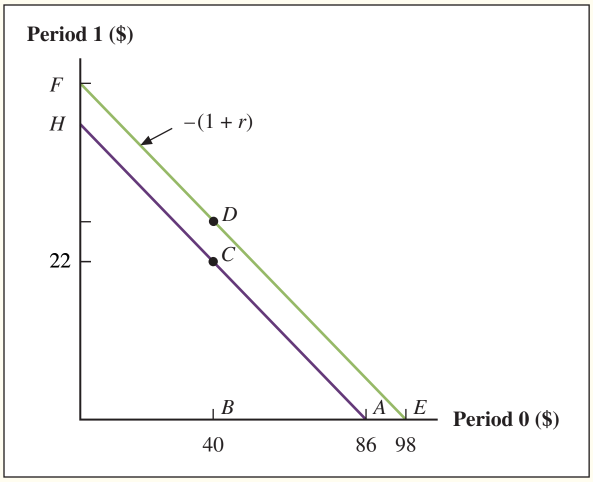
Summary and Conclusions

Finance is a subject that builds understanding from the ground up. Whenever you come up against a new problem or issue in finance, you can always return to the basic principles of this chapter for guidance.

1. Financial markets exist because people want to adjust their consumption over time. They do this by borrowing and lending.
2. Financial markets provide the key test for investment decision making. Whether a particular investment decision should or should not be taken depends only on this test: If there is a superior alternative in the financial markets, the investment should be rejected; if not, the investment is worth making. The most important thing about this principle is that the investor need not use personal preferences to decide whether the investment should be taken. Regardless of the individual's preference for consumption this year versus the next, regardless of how patient or impatient the individual is, making the proper investment decision depends only on comparing it with the alternatives in the financial markets.
3. The net present value of an investment helps us compare the investment and the financial market. If the NPV is positive, our rule tells us to undertake the investment. This illustrates the second major feature of the financial markets and investment. Not only does the NPV rule tell us which investments to accept and which to reject; the financial markets also provide us with the tools for actually acquiring the funds to make the investments. In short, we use the financial markets to decide both what to do and how to do it.
4. The NPV rule can be applied to corporations as well as to individuals. The separation theorem developed in this appendix conveys that all of the owners of the firm would agree that the firm should use the NPV rule even though each might differ in personal tastes for consumption and savings.

Questions and Problems

1. **Consumption Choices** Currently Jim Morris makes $80,000. Next year his income will be $90,000. Jim is a big spender and he wants to consume $150,000 this year. The equilibrium interest rate is 10 percent. What will be Jim's consumption potential next year if he consumes $100,000 this year?
2. **Consumption Choices** Rich Pettit is a miser. His current income is $50,000; next year he will earn $60,000. He plans to consume only $35,000 this year. The current interest rate is 12 percent. What will Rich's consumption potential be next year?
3. **Financial Markets** What is the basic reason that financial markets develop?
4. **Consumption Choices** The following figure depicts the financial situation of Julia Fawn. In period 0 her labor income and current consumption is $40; later, in period 1, her labor income and consumption will be $50. She has an opportunity to make the investment represented by point *D*. By borrowing and lending, she will be able to reach any point along the line *FDE*.
   - **a.** What is the market rate of interest? (*Hint*: The new market interest rate line *EF* is parallel to *AH*.)
   - **b.** What is the NPV of point *D*?
   - **c.** If Julia wishes to consume the same quantity in each period, how much should she consume in period 0?



5. Consumption Choices Harry Hernandez has $60,000 this year. He faces the investment opportunities represented by point *B* in the following figure. He wants to consume $20,000 this year and $67,500 next year. This pattern of consumption is represented by point *F*.

a. What is the market interest rate?

b. How much must Harry invest in financial assets and productive assets today if he follows an optimal strategy?

c. What is the NPV of his investment in nonfinancial assets?